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Stochastic Comparisons of Order Statistics from a Sample Following the Power Lomax Distribution

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Abstract

For independent random variables following the Power Lomax distribution in the two-dimensional case: If two independent sets of Power Lomax samples share the same scale parameter, and the matrix formed by their shape parametersαandβsatisfies a specific chain majorization order, the lifetimes of series systems can be compared under the hazard rate order. Under the same conditions, the lifetimes of parallel systems can be compared under the reverse hazard rate order. These results imply that, under specific constraints, we can systematically evaluate the lifetime properties of different system architectures.

Keywords

Order Statistics; Power Lomax Distribution; Hazard Rate Order; Reverse Hazard Rate Orde.

1. Introduction

The Lomax distribution is a widely utilized probability distribution first proposed by Lomax[1] for modeling business failure data. Rady, Hassanein, et al. [2] proposed and studied a three-parameter continuous distribution, namely the Power Lomax distribution, which was applied to the analysis of remission times for bladder cancer data. As an important generalization of the Lomax distribution, the Power Lomax distribution adjusts the parent distribution's properties via an additional shape parameter, thereby enabling more accurate characterization of diverse types of data. While several authors have investigated the fundamental properties of order statistics from Power Lomax random variables, including their distribution and density functions, relatively little attention has been devoted to the stochastic comparison of such order statistics under different stochastic orders. Addressing this research gap, the present study focuses on investigating the stochastic comparison of order statistics from the Power Lomax distribution under various stochastic orderings.

The definition of the Power Lomax distribution is given below.

If the distribution function of a random variable X is

$$F_X(x)=1-\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\alpha},\quad x>0,\ \alpha,\beta,\lambda>0, \tag{1}$$

Equivalently, the pdf is

$$f_X(x)=\frac{\alpha\beta}{\lambda}x^{\beta-1}\left(1+\frac{x^{\beta}}{\lambda}\right)^{-\alpha-1},\quad x>0,\ \alpha,\beta,\lambda>0, \tag{2}$$

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$\alpha,\beta,\lambda>0$ are constants, then X is said to follow the Power Lomax distribution with parameters α,β,λ. α and β are called shape parameters, and λ is called the scale parameter, denoted as $X\sim PL(X;\alpha,\beta,\lambda)$.

Addressing this research gap, the present study focuses on the stochastic comparisons of order statistics from the Power Lomax distribution under the hazard rate order and reversed hazard rate order.

Definition 1.1

Let X and Y be two random variables. If $r_X(t)\geq r_Y(t)$ holds for all $t\in R$, then the random variable X is said to be smaller than Y in the hazard rate order, denoted as $Y\geq_{hr}X$. The hazard rate function of a random variable X is defined as:

$$r_X(t)=\lim_{\Delta t\to 0}\frac{P(t<X\leq t+\Delta t|X>t)}{\Delta t} \tag{3}$$

For an absolutely continuous random variable X, its hazard rate function can be expressed as:

$$r_X(t)=\frac{f(t)}{\bar{F}(t)}. \tag{4}$$

Definition 1.2

Let X and Y denote two random variables. If $\tilde{r}_X(t)\leq\tilde{r}_Y(t)$ holds for all $t\in R$, then the random variable X is said to be smaller than Y in the hazard rate order, denoted as $Y\geq_{rh}X$. The hazard rate function of a random variable X is defined as:

$$\tilde{r}_X(t)=\lim_{\Delta t\to 0}\frac{P(t-\Delta t<X\leq t|X\leq t)}{\Delta t}. \tag{5}$$

Suppose X is an absolutely continuous random variable, its hazard rate function can be expressed as:

$$\tilde{r}_X(t)=\frac{f(t)}{F(t)}. \tag{6}$$

Lemma 1.3[3] Before presenting the lemma, we first provide the definitions of the sets involved in it. Let

$$P_n=\left\{M(x,y;n):x_i>0,y_i>0,(x_i-x_j)(y_i-y_j)\leq 0,i,j=1,\ldots,n\right\},$$
$$Q_n=\left\{M(x,y;n):x_i\geq 1,y_i>0,(x_i-x_j)(y_i-y_j)\leq 0,i,j=1,\ldots,n\right\}$$

and

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$$R_n = \left\{ M(x, y; n) : x_i > 1, y_i \geq 0, (x_i - x_j)(y_i - y_j) \geq 0, i, j = 1, \ldots, n \right\}.$$

Let $\varphi: R_+^2 \longrightarrow R^+$ be a differentiable function, and let φ_n: $R_+^{2n} \longrightarrow R^+$ be a function. Define the function:

$$\Phi_n(A) = \prod_{i=1}^{n} \varphi(a_{1i}, a_{2i}),$$

then for any $A \in P_2, Q_2, R_2, B = AT_w, \Phi_n(A) \geq \Phi_n(B)$, where T_w is a permutation matrix of T. When $n = 2$, φ_2 satisfies Lemma 1.3.

2. Comparisons under Hazard Rate and Reversed Hazard Rate Orders

2.1 Comparisons under HRO

2.1.1 Theorem 1

Let X_1, X_2 be independent random variables following $PL(X_i; \alpha_i, \beta_i, \lambda)$, and let Y_1, Y_2 be another set of independent random variables following $PL(Y_i; \alpha_i^*, \beta_i^*, \lambda)$. When the values of X_1 and X_2 are sufficiently large, if the matrix $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$ satisfies $\alpha_1 \leq \alpha_2$, $\beta_1 \geq \beta_2$ or $\beta_1 \leq \beta_2$, $\alpha_1 \geq \alpha_2$, then:

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \gg \begin{pmatrix} \alpha_1^* & \alpha_2^* \\ \beta_1^* & \beta_2^* \end{pmatrix} \Rightarrow X_{1:2} \leq_{hr} Y_{1:2}$$

2.1.2 Proof of Theorem 1

First, from Equation (1)-(4), the hazard rate function of $X_{1:2}$ is given by

$$r_{X_{1:2}}(x) = \sum_{i=1}^{2} \frac{\alpha_i \beta_i x_i^{\beta_i - 1}}{(\lambda_i + x_i^{\beta_i})},$$

Taking the partial derivative of the hazard rate function with respect to α_i, we obtain

$$\frac{\partial r_{X_{1:2}}(x, \alpha, \beta)}{\partial \alpha_i} = \frac{\beta_i x_i^{\beta_i - 1}}{\lambda_i + x_i^{\beta_i}} = f(\alpha, \beta),$$

where

$$f(\alpha, \beta) = \frac{\beta x^{\beta - 1}}{\lambda + x^{\beta}}, \quad f(0) = 0.$$

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Then, taking the second derivative of $f\left(\alpha,\beta\right)$ with respect to β , we obtain

$$\frac{\partial^2 f\left(\alpha,\beta\right)}{\partial\beta^2}=\frac{2\lambda^2 x^{\beta-1}\ln x+\lambda^2\beta x^{\beta-1}\left(\ln x\right)^2+2\lambda x^{2\beta-1}\ln x-\lambda\beta x^{2\beta-1}\left(\ln x\right)^2}{\left(\lambda+x^{\beta}\right)^3}=\frac{A\left(x\right)}{B\left(x\right)},$$

where

$$A\left(x\right)=2\lambda^2 x^{\beta-1}\ln x+\lambda^2\beta x^{\beta-1}\left(\ln x\right)^2+2\lambda x^{2\beta-1}\ln x-\lambda\beta x^{2\beta-1}\left(\ln x\right)^2,$$
$$B\left(x\right)=\left(\lambda+x^{\beta}\right)^3,$$

Taking the limit of $B\left(x\right)$, we have

$$\lim_{x\to\infty}B\left(x\right)=+\infty>0,$$

Transforming $A\left(x\right)$ and taking the limit, we get

$$\lim_{x\to\infty}\frac{A\left(x\right)}{\lambda\beta x^{2\beta-1}\left(\ln x\right)^2}=\lim_{x\to\infty}\frac{2\lambda^2 x^{\beta-1}\ln x+\lambda^2\beta x^{\beta-1}\left(\ln x\right)^2+2\lambda x^{2\beta-1}\ln x}{\lambda\beta x^{2\beta-1}\left(\ln x\right)^2}-1$$
$$=\lim_{x\to\infty}C\left(x\right)-1,$$

where

$$C\left(x\right)=\frac{2\lambda^2 x^{\beta-1}\ln x+\lambda^2\beta x^{\beta-1}\left(\ln x\right)^2+2\lambda x^{2\beta-1}\ln x}{\lambda\beta x^{2\beta-1}\left(\ln x\right)^2},$$

Taking the limit of $C\left(x\right)$, we obtain

$$\lim_{x\to\infty}C\left(x\right)=\lim_{x\to\infty}\frac{2\lambda}{\beta x^{\beta}\ln x}+\frac{\lambda}{x^{\beta}}+\frac{2}{\beta\ln x}=0.$$

Since $\lim_{x\to\infty}\lambda\beta x^{2\beta-1}\left(\ln x\right)^2=+\infty$, it follows that

$$\lim_{x\to\infty}A\left(x\right)=\lim_{x\to\infty}\left(C\left(x\right)-1\right)\left(\lambda\beta x^{2\beta-1}\left(\ln x\right)^2\right)=-\infty<0,$$

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Therefore, $\lim_{x\to\infty}\frac{\partial^2 f(\alpha,\beta)}{\partial\beta^2}<0$, which means $f(\beta)$ is a concave function as $x\to+\infty$.

In summary, $f(\alpha,\beta)$ is monotonically increasing with respect to β.

Next, taking the partial derivative of the hazard rate function $r_{x_{1:2}}(x,\alpha,\beta)$ with respect to β_i, we obtain

$$\frac{\partial r_{x_{1:2}}(x,\alpha,\beta)}{\partial\beta_i}=\frac{\alpha(\lambda x^{\beta_i-1}+x^{2\beta_i-1}+\lambda\ln x\cdot\beta_i\cdot x^{\beta_i-1})}{\lambda+x^{\beta_i}}=h(\alpha,\beta),$$

Differentiating $h(\alpha,\beta)$, we have

$$\lim_{x\to\infty}\frac{\partial h(\alpha,\beta)}{\partial\beta}=\lim_{x\to\infty}\frac{\alpha\cdot\partial^2 f(\alpha,\beta)}{\partial\beta^2}<0,$$
$$\lim_{x\to\infty}\frac{\partial h(\alpha,\beta)}{\partial\alpha}=\frac{\lambda x^{\beta_i-1}+x^{2\beta_i-1}+\lambda\ln x\cdot\beta_i\cdot x^{\beta_i-1}}{\lambda+x^{\beta_i}}>0,$$

which implies that $h(\alpha,\beta)$ is monotonically increasing with respect to α and monotonically decreasing with respect to β.

Finally, define the function $\varphi(\alpha,\beta)$ as

$$\varphi(\alpha,\beta)=(\alpha_1-\alpha_2)\left(\frac{\partial r_{x_{1:2}}(x,\alpha,\beta)}{\partial\alpha_1}-\frac{\partial r_{x_{1:2}}(x,\alpha,\beta)}{\partial\alpha_2}\right)$$
$$+(\beta_1-\beta_2)\left(\frac{\partial r_{x_{1:2}}(x,\alpha,\beta)}{\partial\beta_1}-\frac{\partial r_{x_{1:2}}(x,\alpha,\beta)}{\partial\beta_2}\right)$$
$$=(\alpha_1-\alpha_2)\left[f(\alpha_1,\beta_1)-f(\alpha_2,\beta_2)\right]+(\beta_1-\beta_2)\left[h(\alpha_1,\beta_1)-h(\alpha_2,\beta_2)\right],$$

when $\alpha_1\le\alpha_2$, $\beta_1\ge\beta_2$ or $\beta_1\le\beta_2$, $\alpha_1\ge\alpha_2$, we have

$$f(\alpha_1,\beta_1)>f(\alpha_2,\beta_2)\ h(\alpha_1,\beta_1)<h(\alpha_2,\beta_1)<h(\alpha_2,\beta_2)\ \left(f(\alpha_1,\beta_1)<f(\alpha_2,\beta_2)\right)$$
$$\left(h(\alpha_1,\beta_1)>h(\alpha_2,\beta_1)>h(\alpha_2,\beta_2)\right),\text{ hence }\varphi(\alpha,\beta)<0.$$

In summary, by Lemma 2.12, we have

$$\begin{pmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{pmatrix}\gg\begin{pmatrix}\alpha_1^* & \alpha_2^*\\ \beta_1^* & \beta_2^*\end{pmatrix}\Rightarrow r_{X_{1:2}}(\alpha,\beta)\ge r_{Y_{1:2}}(\alpha^*,\beta^*)$$
$$\Rightarrow X_{1:2}\le_{hr}Y_{1:2}.$$

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Therefore, Theorem 1 is proved.

2.2 Comparisons Under RHRO

2.2.1 Theorem 2

Let X_1, X_2 be independent random variables following $PL(X_i;\alpha_i,\beta_i,\lambda)$, and let Y_1, Y_2 be another set of independent random variables following $PL(Y_i;\alpha_i^*,\beta_i^*,\lambda)$. When the values of X_1 and X_2 are sufficiently small, if the matrix $\begin{pmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{pmatrix}$ satisfies $\alpha_1 \le \alpha_2, \beta_1 \ge \beta_2$ or $\beta_1 \le \beta_2$, $\alpha_1 \ge \alpha_2$, , then:

$$\begin{pmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{pmatrix} \gg \begin{pmatrix}\alpha_1^* & \alpha_2^*\\ \beta_1^* & \beta_2^*\end{pmatrix} \Rightarrow X_{2:2} \le_{rh} Y_{2:2}.$$

2.2.2 Proof of Theorem 2

First, from Equation (1)-(2),(5)-(6), the reverse hazard rate function of $X_{2:2}$ is given by

$$\tilde{r}_{X_{2:2}}(x) = \sum_{i=1}^{2} \frac{\alpha_i \beta_i x_i^{\beta_i - 1}}{\left(\lambda_i + x_i^{\beta_i}\right)\left[\left(1+\frac{x_i^{\beta_i}}{\lambda_i}\right)^{\alpha_i} - 1\right]}.$$

Taking the partial derivative of the reverse hazard rate function with respect to α_i ,we obtain

$$\begin{aligned}\frac{\partial \tilde{r}_{X_{2:2}}(x,\alpha,\beta)}{\partial \alpha_i} &= f(\alpha_i,\beta_i) + \alpha_i \frac{\partial f(\alpha_i,\beta_i)}{\partial \alpha_i}\\ &= \frac{\beta_i x^{\beta_i - 1}}{\left(\lambda + x^{\beta_i}\right)\left(\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha_i} - 1\right)}\left[1 - \frac{\alpha_i\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha_i}\ln\left(1+\frac{x^{\beta_i}}{\lambda}\right)}{\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha_i} - 1}\right]\\ &= w(\alpha,\beta),\end{aligned}$$

where

$$f(\alpha_i,\beta_i) = \frac{\beta_i x_i^{\beta_i - 1}}{\left(\lambda_i + x_i^{\beta_i}\right)\left[\left(1+\frac{x_i^{\beta_i}}{\lambda_i}\right)^{\alpha_i} - 1\right]}.$$

Next, taking the partial derivative of $w(\alpha,\beta)\sum_{i=1}^{n}(X_i - \bar{X})^2$ with respect to α we obtain

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$$\frac{\partial w(\alpha,\beta)}{\partial\alpha}=\frac{\beta_i x^{\beta_i-1}}{\left(\lambda+x^{\beta_i}\right)}\cdot\frac{\alpha_i\ln\left(1+\frac{x^{\beta_i}}{\lambda}\right)\left(\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha_i}-\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{3\alpha_i}\right)}{\left(\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha_i}-1\right)^4}<0,$$

which implies that $w(\alpha,\beta)$ is monotonically decreasing with respect to α.

Taking the partial derivative of $w(\alpha,\beta)$ with respect to β, we obtain

$$\frac{\partial w(\alpha,\beta)}{\partial\beta}=\frac{\lambda\left(x^{\beta_i-1}+\beta_i x^{\beta_i-1}\ln x\right)+x^{2\beta_i-1}}{\left(\lambda+x^{\beta_i}\right)^2}\cdot\frac{t^{\alpha_i}-1-\alpha_i t^{\alpha_i}}{\left(t^{\alpha_i}-1\right)^2}$$

$$+\frac{\ln x\cdot\beta_i x^{2\beta_i-1}}{\left(\lambda+x^{\beta_i}\right)}\cdot\frac{\left(-t^{\alpha}-t^{2\alpha}\right)\ln \mathrm{t}+\left(2t^{\alpha}+\alpha t^{2\alpha}\right)\ln^2 t-\alpha t^{2\alpha-1}+\alpha t^{\alpha-1}}{\left(t^{\alpha}-1\right)^3}$$

$$>0,$$

where $t=\left(1+\frac{x^{\beta}}{\lambda}\right)$. Therefore $w(\alpha,\beta)$ is monotonically increasing with respect to β.

Next, taking the partial derivative of the reverse hazard rate function $\tilde{r}_{X_{2:2}}(x)=\sum_{i=1}^{2}\frac{\alpha_i\beta_i x_i^{\beta_i-1}}{\left(\lambda_i+x_i^{\beta_i}\right)\left[\left(1+\frac{x_i^{\beta_i}}{\lambda_i}\right)^{\alpha_i}-1\right]}$ with respect to β_i we obtain

$$\frac{\partial\tilde{\gamma}_{X_{2:2}}(x)}{\partial\beta_i}=\frac{\alpha x^{2\beta_i-1}\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha}+\alpha\lambda x^{\beta_i-1}\left(1+\frac{x\beta_i}{\lambda}\right)^{\alpha}+\alpha\beta_i x^{2\beta_i-1}\ln x\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha}}{\left(x^{\beta_i}+\lambda\right)^2\left(\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha}-1\right)^2}$$

$$+\frac{\alpha\beta_i\lambda x^{\beta_i-1}\ln x\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha}-\alpha\lambda x^{\beta_i-1}-\alpha x^{2\beta_i-1}-\alpha\beta_i x^{2\beta_i-1}\ln x}{\left(x^{\beta_i}+\lambda\right)^2\left(\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha}-1\right)^2}$$

$$+\frac{\alpha\beta_i\lambda x^{\beta_i-1}\ln x-\alpha^2\beta_i\lambda^{-1}x^{2\beta_i-1}\ln x\left(1+\frac{x^{\beta_i}}{\lambda}\right)\alpha}{\left(x^{\beta_i}+\lambda\right)^2\left(\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha}-1\right)^2}$$

$$=m(\alpha,\beta),$$

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We perform appropriate algebraic transformations on its numerator: $m(\alpha,\beta)=\dfrac{A(x)-B(x)}{C(x)}$,

where

$$A(x)=\alpha x^{2\beta_i-1}\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha}+\alpha\lambda x^{\beta_i-1}\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha}+\alpha\beta_i x^{2\beta_i-1}\ln x\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha}$$

$$+\alpha\beta_i\lambda x^{\beta_i-1}\ln x\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha}-\alpha^2\beta_i\lambda^{-1}x^{2\beta_i-1}\ln x\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha},$$

$$B(x)=\alpha\lambda x^{\beta_i-1}+\alpha x^{2\beta_i-1}+\alpha\beta_i x^{2\beta_i-1}\ln x+\alpha\beta_i\lambda x^{\beta_o-1}\ln x$$

$$=\alpha x^{\beta_i-1}\left(\lambda+x^{\beta_i}\right)\left(1+\beta_i\ln x\right),$$

$$C(x)=\left(x^{\beta_i}+\lambda\right)^2\left(\left(1+\frac{x^{\beta_i}}{\lambda}\right)^{\alpha}-1\right)^2,$$

Differentiating $m(\alpha,\beta)$ partially with respect to α, we obtain

$$\frac{\partial m(\alpha,\beta)}{\partial\alpha}=\frac{D(x)\left(E(x)+F(x)+G(x)+H(x)\right)}{I(x)},$$

where

$$D(x)=x^{\beta-1},$$

$$E(x)=\left(1+\frac{x^{\beta}}{\lambda}\right)^{\alpha}\left(x^{\beta}+\lambda\right)\left(\beta\ln x+1\right)\left(\left(\alpha\ln\left(1+\frac{x^{\beta}}{\lambda}\right)+1\right)\right.$$

$$\left.-\alpha\beta\lambda^{-1}x^{\beta}\ln x\left(\alpha\ln\left(1+\frac{x^{\beta}}{\lambda}\right)+2\right)\right),$$

$$F(x)=\left(\lambda+x^{\beta}\right)\left(1+\beta\ln x\right)\left[\left(1+\frac{x^{\beta}}{\lambda}\right)^{\alpha}-1\right],$$

$$G(x)=2\left(1+\frac{x^{\beta}}{\lambda}\right)^{2\alpha}\left(\alpha^2\beta\lambda^{-1}x^{\beta}\ln x-\alpha x^{\beta}+\alpha\lambda+\alpha\beta x^{\beta}\ln x+\alpha\beta\lambda\ln x\right)\ln\left(1+\frac{x^{\beta}}{\lambda}\right),$$

$$H(x)=2\alpha\left(\lambda+x^{\beta}\right)\left(1+\beta\ln x\right)\left(1+\frac{x^{\beta}}{\lambda}\right)^{\alpha}\ln\left(1+\frac{x^{\beta}}{\lambda}\right),$$

$$I(x)=\left(x^{\beta}+\lambda\right)^2\left(\left(1+\frac{x^{\beta}}{\lambda}\right)^{\alpha}-1\right)^3,$$

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First, rearrange E :

$$E(x)=\left(1+\frac{x^{\beta}}{\lambda}\right)^{\alpha}\left(x^{\beta}+\lambda\right)\left(\beta\ln x+1\right)\cdot L(x),$$

where

$$L(x)=\left(\alpha\ln\left(1+\frac{x^{\beta}}{\lambda}\right)+1\right)-\alpha\beta\lambda^{-1}x^{\beta}\ln x\left(\alpha\ln\left(1+\frac{x^{\beta}}{\lambda}\right)+2\right).$$

Since

$$\lim_{x\to0^{+}}\beta\ln x+1\to-\infty,\quad \lim_{x\to0^{+}}\left(-\beta\ln x\right)\to+\infty,$$
$$\lim_{x\to0^{+}}\left(x^{\beta}+\lambda\right)\left[\left(1+\frac{x^{\beta}}{\lambda}\right)^{\alpha}-1\right]>0,$$

Therefore, we obtain $\lim_{x\to0^{+}}D(x)>0$, $\lim_{x\to0^{+}}F(x)<0$, $\lim_{x\to0^{+}}G(x)<0$, $\lim_{x\to0^{+}}H(x)<0$.

From $\lim_{x\to0^{+}}x^{\beta-1}>0$ and $\lim_{x\to0^{+}}\left(x^{\beta}+\lambda\right)^{2}\left[\left(1+\frac{x^{\beta}}{\lambda}\right)^{\alpha}-1\right]^{3}>0$, we get $\lim_{x\to0^{+}}D(x)>0$.

Since $\lim_{x\to0^{+}}L(x)>0$, it follows that $\frac{\partial m(x,\alpha,\beta)}{\partial\alpha}=\frac{D(x)\left(E(x)+F(x)+G(x)+H(x)\right)}{I(x)}<0$, which means $m(\alpha,\beta)$ is monotonically decreasing with respect to α.

Next, aking the partial derivative of $m(\alpha,\beta)$ partially with respect to β, we have

$$\frac{\partial m(\alpha,\beta)}{\partial\beta}=\frac{T(x)\left(Q(x)+R(x)+S(x)\right)}{N(x)},$$

where

$$T(x)=x^{\beta-1}\ln x,$$
$$Q(x)=\left(1+\frac{x^{\beta}}{\lambda}\right)^{\alpha}\left[\alpha^{2}\beta x^{2\beta}\ln x+2\lambda x^{\beta}+3\alpha\lambda x^{\beta}\right.$$
$$\left.+2\alpha x^{\beta}\left(x^{\beta}+\lambda\right)-4\lambda\left(x^{\beta}+\lambda\right)-2\lambda^{2}\beta\ln x\right],$$

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$$R(x)=\left(1+\frac{x^{\beta}}{\lambda}\right)^{2\alpha}\left[2\lambda\left(x^{\beta}+\lambda\right)+\lambda^{2}\beta\ln x+\alpha^{2}\beta x^{2\beta}\ln x\right.$$
$$\left.-\lambda x^{\beta}-3\alpha\lambda x^{\beta}-2\alpha x^{\beta}\left(x^{\beta}+\lambda\right)\right],$$
$$S(x)=2\lambda\left(x^{\beta}+\lambda\right)+\beta\lambda^{2}\ln x-\lambda x^{\beta},$$
$$N(x)=\left(x^{\beta}+\lambda\right)\left[\left(1+\frac{x^{\beta}}{\lambda}\right)^{\alpha}-1\right]^{3},$$

Rearranging $Q(x)$, we obtain

$$Q(x)=\left(1+\frac{x^{\beta}}{\lambda}\right)^{\alpha}\cdot N(x),$$

where

$$N(x)=\alpha^{2}\beta x^{2\beta}\ln x+2\lambda x^{\beta}+3\alpha\lambda x^{\beta}+2\alpha x^{\beta}\left(x^{\beta}+\lambda\right)-4\lambda\left(x^{\beta}+\lambda\right)-2\lambda^{2}\beta\ln x$$
$$=2\alpha\mathrm{x}^{2\beta}+5\alpha\lambda x^{\beta}+\alpha^{2}\beta x^{2\beta}\ln x-2\lambda x^{\beta}-2\beta\lambda^{2}\ln x-4\lambda^{2}=\eta(x)-\omega(x),$$
$$\eta(x)=2\alpha x^{2\beta}+5\alpha\lambda x^{\beta}+\alpha^{2}\beta x^{2\beta}\ln x, \text{ and}$$
$$\omega(x)=2\lambda x^{\beta}+2\beta\lambda^{2}\ln x+4\lambda^{2},$$
$$\frac{\eta(x)}{\omega(x)}=\frac{\alpha x^{\beta}}{2\lambda}\cdot\frac{5\lambda+2x^{\beta}+\alpha\beta x^{\beta}\ln x}{2\lambda+x^{\beta}+\lambda\beta\ln x}=\frac{\alpha x^{\beta}}{2\lambda}\cdot\frac{\phi(x)}{\psi(x)},$$
$$\phi(x)=5\lambda+2x^{\beta}+\alpha\beta x^{\beta}\ln x,\ \psi(x)=2\lambda+x^{\beta}+\lambda\beta\ln x,$$

then

$$\frac{\phi(x)}{\psi(x)}=1+\frac{3\lambda+x^{\beta}+\beta\ln x\left(\alpha x^{\beta}-\lambda\right)}{\psi(x)}.$$

As $x\to 0^{+}$, we have

$$\lim_{x\to0^{+}}\beta\ln x\to-\infty,\ \lim_{x\to0^{+}}\alpha x^{\beta}-\lambda<0,\ \lim_{x\to0^{+}}\psi(x)<0,$$

which leads to the following results:

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$$0 < \lim_{x \to 0+} \frac{\phi(x)}{\psi(x)} < 1,$$

$$\eta(x) - \omega(x) < 0,$$

$$\lim_{x \to 0+} Q(x) < 0.$$

Next, we rearrange $R(x)$

$$\begin{aligned} R(x) = \left(1 + \frac{x^\beta}{\lambda}\right)^{2\alpha} \Big[& 2\lambda\left(x^\beta + \lambda\right) + \lambda^2 \beta \ln x + \alpha^2 \beta x^{2\beta} \ln x \\ & -\lambda x^\beta - 3\alpha\lambda x^\beta - 2\alpha x^\beta \left(x^\beta + \lambda\right) \\ = & 2\lambda^2 + \lambda x^\beta + \lambda^2 \beta \ln x + \alpha^2 \beta x^{2\beta} \ln x - 5\alpha\lambda x^\beta - 2\alpha x^{2\beta} \\ = & \mu(x) - \nu(x), \end{aligned}$$

where

$$\mu(x) = 2\lambda^2 + \lambda x^\beta + \lambda^2 \beta \ln x + \alpha^2 \beta x^{2\beta} \ln x,$$

$$v(x) = 5\alpha\lambda x^\beta + 2\alpha x^{2\beta},$$

then

$$\frac{\mu(x)}{v(x)} = \frac{1}{\alpha} \cdot \frac{2\lambda^2 + \lambda x^\beta + \lambda^2 \beta \ln x + \alpha^2 \beta x^{2\beta} \ln x}{5\lambda x^\beta + 2x^{2\beta}} = \frac{1}{\alpha} \cdot \frac{P(x)}{Z(x)},$$

where

$$P(x) = 2\lambda^2 + \lambda x^\beta + \lambda^2 \beta \ln x + \alpha^2 \beta x^{2\beta} \ln x,$$

$$Z(x) = 5\lambda x^\beta + 2x^{2\beta},$$

hence

$$P(x) - Z(x) = \lambda\left(2\lambda + \lambda\beta \ln x - 4x^\beta\right) + x^{2\beta}\left(\alpha^2 \beta \ln x - 2\right),$$

when $x \to 0^+$

$$\lim_{x \to 0+} P(x) - Z(x) < 0.$$

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So

$$\frac{\mu(x)}{\nu(x)}<1,$$

$$\lim_{x\to 0^+} F'(x)=\lim_{x\to 0^+}\mu(x)-\nu(x)<0.$$

Furthermore, as $x \to 0^+$,

$$\lim_{x\to 0^+} S(x)=\lim_{x\to 0^+} 2\lambda\left(x^\beta+\lambda\right)+\beta\lambda^2\ln x-\lambda x^\beta$$
$$=2\lambda^2+\lambda x^\beta+\beta\lambda^2\ln x\to-\infty,$$

thus $\lim_{x\to 0^+} S(x)<0$.

In summary, we obtain

$$\lim_{x\to 0^+} T(x)=\lim_{x\to 0^+} x^{\beta-1}\ln x<0, \lim_{x\to 0^+} N(x)$$
$$=\lim_{x\to 0^+}\left(x^\beta+\lambda\right)\left[\left(1+\frac{x^\beta}{\lambda}\right)^\alpha-1\right]^3>0$$

And

$$\lim_{x\to 0^+} Q(x)<0,\ \lim_{x\to 0^+} R(x)<0,\ \lim_{x\to 0^+} S(x)<0,$$

Therefore, we have

$$\frac{\partial m(x,\alpha,\beta)}{\partial\beta}=\frac{T(x)\cdot\left(\mathrm{Q}(x)+\mathrm{R}(x)+\mathrm{S}(x)\right)}{N(x)}>0,$$

which implies that $m(x,\alpha,\beta)$ is monotonically increasing with respect to β.

Finally, define the function $\varphi(\alpha,\beta)$ as

$$\varphi(\alpha,\beta)=(\alpha_1-\alpha_2)\left(\frac{\partial r_{x_{2:2}}(x,\alpha,\beta)}{\partial\alpha_1}-\frac{\partial r_{x_{2:2}}(x,\alpha,\beta)}{\partial\alpha_2}\right)$$
$$+(\beta_1-\beta_2)\left(\frac{\partial r_{x_{2:2}}(x,\alpha,\beta)}{\partial\beta_1}-\frac{\partial r_{x_{2:2}}(x,\alpha,\beta)}{\partial\beta_2}\right)$$
$$=(\alpha_1-\alpha_2)\left[w(\alpha_1,\beta_1)-w(\alpha_2,\beta_2)\right]$$

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$$+(\beta_1-\beta_2)\left[m(\alpha_1,\beta_1)-m(\alpha_2,\beta_2)\right],$$

when $\alpha_1 \le \alpha_2$, $\beta_1 \ge \beta_2$ or $\beta_1 \le \beta_2, \alpha_1 \ge \alpha_2$, we have

$$w(\alpha_1,\beta_1) > w(\alpha_2,\beta_2)\left(w(\alpha_1,\beta_1) < w(\alpha_2,\beta_2)\right)$$

and

$$m(\alpha_1,\beta_1) < m(\alpha_2,\beta_2) < m(\alpha_2,\beta_2)\left(m(\alpha_1,\beta_1) > m(\alpha_2,\beta_1) > m(\alpha_2,\beta_2)\right),$$

hence $\varphi(\alpha,\beta) < 0$.

In summary, by Lemma 1.3, we obtain the following conclusion,

$$\begin{pmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{pmatrix} \gg \begin{pmatrix}\alpha_1^* & \alpha_2^*\\ \beta_1^* & \beta_2^*\end{pmatrix} \Rightarrow r_{X_{2:2}}(x,\alpha,\beta) \le r_{X_{2:2}^*}(x,\alpha,\beta)$$
$$\Rightarrow X_{2:2} \le_{rh} Y_{2:2}.$$

Therefore, Theorem 2 is proved.

3. Summary

For independent two-dimensional random variables following the Power Lomax distribution, the process of comparing system lifetimes is structured as follows: first, ensure two independent sets of samples share the same scale parameter; second, verify that the matrix composed of shape parameters α and β satisfies a specific chain majorization order; based on these prerequisites, conduct targeted comparisons-for series systems, adopt the hazard rate order as the evaluation criterion, while for parallel systems, use the reverse hazard rate order; finally, through this systematic framework, achieve quantitative assessment of the lifetime properties of different system architectures under specific constraints.

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