

The Enumeration Formula of Standard Young Tableaux for Approximate Right Triangles

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Abstract

The correspondence between nested order statistics and standard Young tableaux reveals a profound link between probability theory and combinatorics. By interpreting standard Young tableaux as nested order statistics constructions, the enumeration problem transforms into computing multiple integrals over simplex domains constrained by nested order statistics. This probabilistic-combinatorial duality enables analytical techniques from integral geometry to estimate standard Young tableaux counts. Otherwise, we combine with the combinatorial identity, the general summation representation of hollow shifted standard Young tableaux with approximately right-angled trapezoids is given, thus verifying the conjecture in the literature.

Keywords

Standard Young Tableaux; Hollow Shifted Shapes; Multiple Integral; Catalan Number.

1. Introduction

Standard Young tableaux (SYT) have a wide range of applications in geometry as well as in substitution groups of abstract algebra and irreducible representations of general linear groups [1-2]. Many scholars have studied the counting of different standard Young tableaux with different methods. Recent combinatorial investigations have focused on the enumeration of truncated hollow standard Young tableaux (SYTs). The foundational framework for truncated SYTs was first established by G. D. James and M. H. Peel [3], who introduced the geometric construction and pivotal combinatorial properties. Building on this work, subsequent research in [4] proposed a novel algebraic interpretation of truncated SYTs through representation-theoretic methods, resolving key conjectures in the field. R.M. Adin and Y. Roichman systematically investigated the enumeration of standard Young tableaux (SYTs) with northeastern corner truncations [5]. Building upon this geometric configuration analysis, Adin et al. established a combinatorial framework employing pivoting theory to derive closed-form expressions for square-truncated rectangular and shifted staircase-shaped SYTs, with particular emphasis on near-square truncation patterns. G. Panova [6] established explicit formulas for square-truncated rectangular Young diagrams with staircase approximations through Schur function decompositions in symmetric function theory. Building on this algebraic combinatorial framework, P. Sun and Li [7-11] developed a probabilistic-analytic framework demonstrating that SYT enumerations for both regular and shifted truncated shapes emerge as specializations of nested sequence statistic distributions (see Theorem 3.2 in [7]), thereby establishing systematic connections between tableau enumeration and permutation pattern analysis.

A partition λ [12] of a positive integer n is defined as a weakly decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_d)$ where $n = \lambda_1 + \dots + \lambda_d$. The associated Young diagram, a canonical visual representation of λ , consists of left-justified rows of boxes with λ_i boxes in the i -th row (indexed top-

to-bottom). Here λ_j denotes the conjugate partition specifying λ_j boxes in the j -th column (indexed left-to-right).

In [13], Adin and Roichman rigorously characterized standard Young tableaux (SYTs) as bijective labelings of Young diagrams where $\{1, 2, \dots, n\}$ entries strictly increase along rows (left-to-right) and columns (top-to-bottom). This combinatorial construct extends the classical hook-length formula – the seminal enumerative result for SYT counts – through innovative geometric-algebraic correspondence techniques applied to tableau configurations. A SYT of rectangular shape is denoted by $n^m = (n, n, \dots, n)$ and it is clear that the SYT of truncated shapes can be represented by $\lambda \setminus \mu$ or λ / ν , namely λ, μ, ν are partitions of positive integers. Instead of deleting μ_i cells from the end of row i , a truncated shape $\lambda \setminus \mu$ is a left-justified array of cells which belong to λ rather than ν and the shape λ / ν is the cells which belong to λ rather than ν when cells in the upper left corners are kicked out [7].

The SYT of a hollow truncated shape is denoted by $\lambda \setminus \mu | \{(i_0, j_0)\}$ that the diagram of shape λ deletes the cells belonging to shape μ from (i_0, j_0) , while the cells of μ in the upper left corners are deleted in row i_0 and column j_0 of λ . The SYT of truncated shape $\lambda \setminus \mu | \{(i_0, j_0)\}$ is filled by a corresponding truncated diagram with the integers from 1 to $|\lambda| - |\mu|$ so that each row and column is increasing in [14].

For example, $\delta_{n+2}(H), (n \geq 2)$ is denoted as Figure 1.

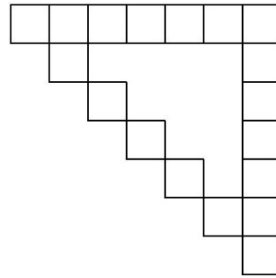


Fig.1 SYT of the $\delta_{n+2}(H), (n \geq 2)$ shape

P. Sun [14] established recursive formulae for SYT counts on nearly hollow rectangular diagrams while formalizing axiomatic definitions of hollow SYTs. Contemporary research extends this paradigm to exotic diagram topologies – polygonal and alphabetic Young diagrams have emerged as active frontiers. Specifically, Standard Young Tableaux have new progress in Dyck path and experimenting[15-16]. Building upon this progression, we derive enumeration formulae for Approximate Right Triangles as Fig. 1.

In this paper, the counting problem of standard Young tableaux is transformed into a multiple integration problem on the corresponding nested order statistics[7], Through the calculation of multiple integrals on the distribution $U(0,1)$ of nested order statistics, the number of standard Young tableaux of λ shape is obtained as

$$H_\lambda = |\lambda|! \text{Vol}(S_\lambda) = |\lambda|! \int_{S_\lambda} \dots \int 1 dx_{1, \lambda_1} \dots dz_{\lambda_d, \lambda_d} \quad (1)$$

Where $\text{Vol}(S_\lambda)$ is the volume of the nested simplex S_λ corresponding to the standard Young tableaux of type λ , i.e., the domain of integration.

2. Main Results

In this paper, according to the correspondence between standard Young tableaux and nested order statistics, we mainly focus on transforming the counting problem of standard Young tableaux into a multiple integration problem on the model of nested order statistics. This section focuses on the precise enumeration of standard Young tableaux of shape $\delta_{n+2}(H), (n \geq 2)$ (as shown in Figure 1) through the partitioning and analysis of nested simplex. By leveraging computational methods from [7], we rigorously calculate the combinatorial count of such tableaux, simplify the results using relevant combinatorial identities, and ultimately derive a closed-form enumeration formula. This resolves the conjecture proposed in [14], thereby confirming its validity.

Theorem 1. The enumeration formula for standard Young tableaux of shape $\delta_{n+2}(H), (n \geq 2)$ is

$$H_{\delta_{n+2}(H), (n \geq 2)} = \frac{(3n+1)!}{n!(2n+3)!} (n^2 + 3n + 6) \quad (2)$$

The result of this counting formula is the Catalan number 6th convolution [17.A143388].

As n varies, the numbers of SYT-type charts of $\delta_{n+2}(H), (n \geq 2)$ shape are listed as follows:

Table 1. SYT-type chart counts for shape $\delta_{n+2}(H), (n \geq 2)$

n	1	2	3	4	5	6	7	8	
$H_{\delta_{n+2}(H), (n \geq 2)}$	1	2	8	40	221	1288	7752	47652	

Proof. From Lemma 1, the counting formula for the standard Young tableaux of this shape should be

$$H_{\delta_{n+2}(H), (n \geq 2)} = (3n+3)! I_n = (3n+3)! \int_{D_n} \cdots \int dx_r dy_r dt_r \quad (3)$$

where the integral range D_n represents the nested simplex corresponding to the standard Young tableaux of shape $\delta_{n+2}(H), (n \geq 2)$, and the variables must satisfy the strictly increasing from left to right and from top to bottom in each row and column, respectively. that is

$$D_n = \left\{ \begin{array}{ccccccc} 0 < x_0 < x_1 < x_2 < \cdots < x_n < x & & & & & & \\ & \wedge & \wedge & \wedge & \wedge & \wedge & \\ & & y_1 & & & & t_1 \\ & & & \wedge & & & \wedge \\ & & & & y_2 < & & t_2 \\ & & & & & \wedge & \vdots \\ & & & & & \ddots & \vdots \\ & & & & & \wedge & \wedge \\ & & & & & & y_n < t_n \\ & & & & & & \wedge \\ & & & & & & t < 1 \end{array} \right\}$$

The integration domain here is relatively complex and requires detailed analysis. The specific steps and rationale are as follows:

Step 1: Deformation analysis of SYT-type charts of $\delta_{n+2}(H), (n \geq 2)$ shape

We first discuss the possible values of the variable x . There are $n+1$ distinct cases: $y_{j-1} < x < y_j, (j = 2, 3, \dots, n)$, $x_n < x < y_1$ and $y_n < x < t_1$. Consequently, SYT-type charts of type $\delta_{n+2}(H), (n \geq 2)$ are partitioned into three groups for systematic analysis.

Step 2: Integral computations for each group.

For each group of SYT-type charts, we perform combinatorial integration to derive their respective enumeration formulas. By leveraging the equivalence between pre- and post-deformation counting problems, we simplify the computation to enumerating only the deformed Young tableaux. Critical intermediate results from relevant lemmas are invoked during this process.

Step 3: Summation and simplification.

The results from the three groups are aggregated. Utilizing combinatorial identities, we unify the piecewise expressions into a closed-form formula for the number of SYT-type charts of type $\delta_{n+2}(H), (n \geq 2)$.

Thus, the original domain D_n is decomposed into three components $D_1(j), D_2, D_3$ for multivariate integral computations.

$$\begin{aligned}
 & 0 < x_0 < x_1 < \dots < x_{j-1} < x_j < \dots < x_n \\
 & \quad \wedge \quad \wedge \quad \wedge \quad \quad \quad \wedge \\
 & y_1 < \dots < y_{j-1} \quad \quad \quad < x \quad \quad \quad < y_j < \dots < y_n \\
 & \quad \quad \quad \wedge \quad \quad \quad \wedge \quad \wedge \quad \wedge \\
 & \quad \quad \quad t_1 < \dots < t_{j-1} < t_j < \dots < t_n < t < 1 \\
 & D_1(j), 2 \leq j \leq n, \\
 & 0 < x_0 < x_1 < \dots < x_n < x < y_1 < y_2 < \dots < y_n \\
 & \quad \quad \quad \wedge \quad \wedge \quad \wedge \quad \wedge \\
 & \quad \quad \quad t_1 < t_2 < \dots < t_n < t < 1 \\
 & D_2 \\
 & 0 < x_0 < x_1 < \dots < x_{n-1} < x_n \\
 & \quad \quad \quad \wedge \quad \wedge \quad \wedge \quad \wedge \\
 & y_1 < \dots < y_{n-1} < y_n < x < t_1 < t_2 < \dots < t_{n-1} < t_n < t < 1 \\
 & D_3
 \end{aligned}$$

Let

$$I_n = \sum_{j=2}^n I_1(j) + I_2 + I_3, \quad (4)$$

Included among these

$$\begin{aligned}
 I_1(j) &= \int_{D_1(j)} \dots \int dx_r dy_r dt_r, \\
 I_2 &= \int_{D_2} \dots \int dx_r dy_r dt_r, \\
 I_3 &= \int_{D_3} \dots \int dx_r dy_r dt_r.
 \end{aligned}$$

when $2 \leq j \leq n$ is calculated by integrating

$$I_1(j) = \int \cdots \int_{D_1^*} \frac{(x - x_{j-1})^{n-j+1}}{(n-j+1)!} \cdot \frac{(t_j - x)^{j-1}}{(j-1)!} dy_r dt_r dx_r dx_0 dt, \quad (5)$$

Where

$$D_1^* = \left\{ \begin{array}{l} 0 < x_0 < x_1 < \cdots < x_{j-1} \\ \quad \wedge \quad \wedge \quad \wedge \\ y_1 < \cdots < y_{j-1} < x < y_j < \cdots < y_n \\ \quad \quad \quad \wedge \quad \wedge \quad \wedge \\ \quad \quad \quad t_j < \cdots < t_n < t < 1 \end{array} \right\}.$$

To simplify the calculation, do variable substitution

$$\begin{cases} x_r = x_0 + (x - x_0)u_r, 1 \leq r \leq j-1 \\ y_r = x_0 + (x - x_0)v_r, 1 \leq r \leq j-1 \\ y_s = x + (t - x)v_s, j \leq s \leq n \\ t_s = x + (t - x)w_s, j \leq s \leq n \end{cases}$$

We obtain

$$I_1(j) = \int \cdots \int_{D_1^{**}} \frac{(x - x_0)^{n-j+1} (1 - u_{j-1})^{n-j+1}}{(n-j+1)!} \cdot \frac{(t - x)^{j-1} w_j^{i-1}}{(j-1)!} \cdot (x - x_0)^{2j-2} (t - x)^{2n-2j+2} du_r dv_r dv_s dw_s dx dx_0 dt \quad (6)$$

Where

$$D_1^{**} = \left\{ \begin{array}{l} 0 < u_1 < u_2 < \cdots < u_{j-1} \quad 0 < v_j < v_{j+1} < \cdots < v_n \\ \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \text{and } 0 < x_0 < x < t < 1 \\ v_1 < v_2 < \cdots < v_{j-1} < 1, \quad w_j < w_{j+1} < \cdots < w_n < 1 \end{array} \right\}.$$

Combined with Euler's formula

$$\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

We have

$$I_1(j) = \int \cdots \int_{D_1^{**}} \frac{(x - x_0)^{n-j+1} (1 - u_{j-1})^{n-j+1}}{(n-j+1)!} \cdot \frac{(t - x)^{2n-j+1} w_j^{i-1}}{(j-1)!} du_r dv_r dv_s dw_s dx dx_0 dt$$

$$I_1(j) = \frac{\Gamma(n+j+1)\Gamma(2n-j+3)}{\Gamma(3n+4)(n+j)(2n-j+2)} \times I_1^*(j) = \frac{(n+j-1)!(2n-j+1)!}{(3n+3)!} \times I_1^*(j) \quad (7)$$

Where

$$I_1^*(j) = \int \cdots \int_{D_1^{**}} \frac{w_j^{j-1}}{(j-1)!} \frac{(1-u_{j-1})^{n-j+1}}{(n-j+1)!} du_r dv_r dv_s dw_s$$

Calculated with the aid of the determinant integral method

$$I_1^*(j) = \frac{(n-j+2)}{(n+1)!(j-1)!} \left[\frac{1}{n!(n-j+1)!} - \frac{1}{(n-j)!(n+1)!} \right] \quad (8)$$

Therefore, substituting Eq. (8) into Eq. (7) yields

$$I_1(j) = \frac{1}{(3n+3)!} \left[\frac{1}{(n+1)} \binom{n+j-1}{j-1} \binom{2n-j+1}{n-j+1} + \binom{n+j-1}{j-2} \binom{2n-j+1}{n-j} \right] \quad (9)$$

The same reasoning leads to

$$I_2 = \frac{1}{(3n+3)!} \frac{(2n)!}{(n+1)!n!}, \quad I_3 = \frac{1}{(3n+3)!} \frac{(2n)!}{(n+1)!n!} \quad (10)$$

Combined with Eq. (9)-(10), it can be shown that

$$\sum_{j=2}^n I_1(j) = \sum_{j=2}^n \frac{1}{(3n+3)!} \left[\frac{1}{(n+1)} \binom{n+j-1}{j-1} \binom{2n-j+1}{n-j+1} + \binom{n+j-1}{j-2} \binom{2n-j+1}{n-j} \right] \quad (11)$$

Summarizing, we can obtain

$$H_{\delta_{n+2}(H), (n \geq 2)} = \sum_{j=2}^n \frac{1}{(n+1)} \binom{n+j-1}{j-1} \binom{2n-j+1}{n-j+1} + \sum_{j=0}^{n+1} \binom{n+j-1}{j-2} \binom{2n-j+1}{n-j} + I_2 + I_3 \quad (12)$$

Sign

$$H_{\delta_{n+2}(H), (n \geq 2)} = S_1 + S_2 + I_2 + I_3$$

By the combinatorial constant equation

$$\sum_{i=0}^n \binom{x+k}{k} \binom{y+n-k}{n-k} = \binom{x+y+n+1}{n}$$

We have

$$S_1 = \frac{1}{(n+1)} \binom{3n+1}{n} - \frac{2}{(n+1)} \binom{2n}{n} \quad (13)$$

$$S_2 = \binom{3n+1}{n-2} \quad (14)$$

And the result is

$$H_{\delta_{n+2}(H), (n \geq 2)} = \frac{(3n+1)!}{n!(2n+3)!} (n^2 + 3n + 6) \quad (15)$$

It is the 6 th convolution of Catalan number ^[17, A143388], which shows that theorem1 holds, and thus the conjecture in the literature[14] is prove.

Theorem 2 ^[14. Conjecture]. For $n \geq 2$, the enumeration formula for standard Young tableaux of shape $((n+2)^{(n+2)}) \setminus \delta_{n+1}(H), (n \geq 2)$ (as illustrated in Fig.3(b)) is:

$$H_{((n+2)^{(n+2)}) \setminus \delta_{n+1}(H), (n \geq 2)} = \frac{(3n+1)!}{n!(2n+3)!} (n^2 + 3n + 6) \quad (16)$$

which corresponds to a convolution of the Catalan triangle ^[17, A143388].

Proof. The labeling principle for standard Young tableaux requires that entries strictly increase across each row and down each column. The staircase-shaped standard Young tableaux of types $\delta_{n+2}(H), (n \geq 2)$ and $((n+2)^{(n+2)}) \setminus \delta_{n+1}(H), (n \geq 2)$ are illustrated in Fig. 2(a) and Fig.2(b), respectively.

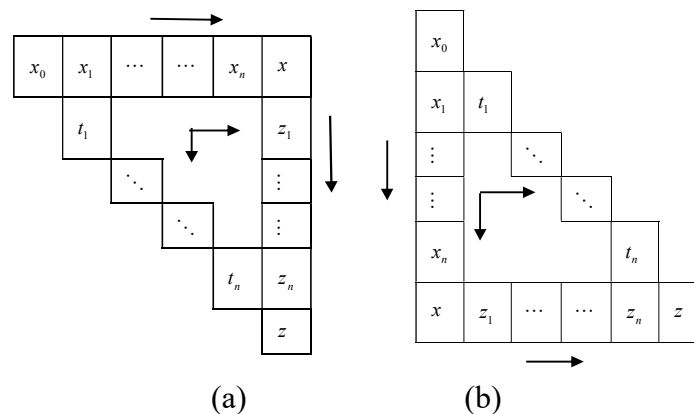


Fig.2 SYT of the $\delta_{n+2}(H), (n \geq 2)$ shape and the $((n+2)^{(n+2)}) \setminus \delta_{n+1}(H)$ shape

The direction indicated by the arrow corresponds to the increasing order of the variable. It is evident that the integration variable ranges for both classes of standard Young tableaux are identical; thus, their derived formulas should coincide. Therefore, based on the result of Theorem 1, the enumeration formula for the shifted staircase-shaped standard Young tableaux of type $((n+2)^{(n+2)}) \setminus \delta_{n+1}(H), (n \geq 2)$ is:

$$H_{((n+2)^{(n+2)}) \setminus \delta_{n+1}(H), (n \geq 2)} = \frac{(3n+1)!}{n!(2n+3)!} (n^2 + 3n + 6)$$

which represents a convolution of the Catalan triangle ^[17, A143388].

3. Conclusion

This paper transforms the counting problem of Standard Young Tableaux into multiple integrals over nested order statistics, thereby obtaining the enumeration of two classes of Standard Young Tableaux with approximately right-angled trapezoidal shapes. This confirms the conjecture in Reference, demonstrating that the result corresponds to a convolution of Catalan numbers. The work provides new theoretical support for the study of Standard Young Tableaux enumeration and holds significant importance in advancing combinatorial research.

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