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A Generalization of the Marshall-Olkin Bivariate Pareto Distribution and its Parameter Estimation

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Abstract

As a generalization of the Marshall-Olkin bivariate Pareto distribution proposed by Kumar Dey and Paul, we introduce the proportional reversed hazard rate Marshall-Olkin bivariate Pareto distribution (PRH-MOBVPA) and the Block-Basu proportional reversed hazard rate bivariate Pareto distribution (PRH-BBBVPA). This paper investigates the relevant probabilistic properties of these two distributions, including expressions for their joint probability density function, joint survival function, probability density functions of marginal distributions and conditional distributions. Additionally, by comparing surface plots and contour plots of the joint density function of the PRH-BBBVPA distribution under different parameters, we reveal how its joint density function varies with different parameters. Furthermore, the maximum likelihood estimators of the six parameters of the PRH-MOBVPA distribution are derived via the expectation-maximization (EM) algorithm.

Keywords

Marshall-Olkin Bivariate Pareto Distribution; Proportional Reversed Hazard Rate Parameter; Maximum Likelihood Estimators; EM Algorithm.

1. Introduction

1.1 Research Background and Significance

In the field of multivariate data analysis, the multivariate normal distribution has become a primary research focus for many scholars due to its concise mathematical form and rich statistical theory. However, in practical research, including financial risk management, insurance, and environmental sciences, it has been gradually observed that datasets in these fields often exhibit tail dependence (i.e., extreme events are not mutually independent) as well as heavy-tailed properties. Against this backdrop, scholars have initiated in-depth investigations into multivariate Pareto distributions, exploring their different types and characteristics. For instance, some multivariate Pareto distributions emphasize capturing linear dependencies among variables, while others do well in modeling heavy-tailed properties. Additionally, certain distributions enhance their fitting capabilities for extreme values by incorporating specific parameters. Among multivariate Pareto distributions, the Marshall-Olkin bivariate Pareto distribution (MOBVPA) has demonstrated certain advantages in relevant applications due to its precise modeling capability for bivariate dependence structures. However, MOBVPA has some limitations: it only includes location, scale, and proportional hazard rate parameters, which limit its flexibility in fitting diverse datasets and make it challenging to address the varied tail characteristics and dependencies flexibly encountered in real-world scenarios.

Therefore, this study focuses on extending the MOBVPA. By introducing new parameters, we aim to enhance the model's fitting capability for complex real-world data, thereby increasing its value in application to extreme event analysis and risk management in finance, insurance, and environmental

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sectors. This extension provides more flexible and reliable statistical tools for empirical research in these fields.

1.2 Current State of Domestic and International Research

(1) Domestic research status

Regarding the univariate Pareto distribution, Yang Meng et al. applied the generalized Pareto distribution model to analyze extreme precipitation events in the Chengdu Economic Zone and estimated its parameters[1]. Mou Tingting et al. used the generalized Pareto distribution to fit extreme precipitation data. They estimated the return periods of extreme precipitation events at different return levels[2]. Zhao Ruixing et al. proposed the Pickands bootstrap moment estimation method to address parameter estimation issues for the generalized Pareto distribution. They applied this distribution to the field of extreme precipitation[3]. Yang Yonghong et al. conducted in-depth research on the exponential Pareto distribution, examining the convergence, asymptotic distribution, and numerical characteristics of the maximum order statistic under conditions such as limited sample sizes and varying parameter values[4]. Regarding the bivariate Pareto distribution, Li Guoan et al. employed multiple statistical methods, including maximum likelihood estimation and moment estimation, to estimate its parameters[5].

(2) Current status of international research

Over the past several decades, the Pareto distribution has emerged as a core tool in the analysising and modeling of the probability characteristics of extreme events due to its distinctive heavy-tailed properties[6-9]. In studies of the univariate Pareto distribution, researchers such as Arnold have employed Bayesian analysis methods to estimate its parameters and applied these estimates to income level forecasting[10-11].The univariate Pareto distribution has played a significant role in the analysis of income and wealth data; the bivariate Pareto distribution has demonstrated broad application potential in numerous fields including finance, failure time prediction, and income and wealth modeling; while the multivariate Pareto distribution is specifically used to construct dependent risk models closely related to financial business lines[12]. It is also employed in stress-strength models to assess system reliability[13]. Within the field of multivariate Pareto distribution research, Mardia first proposed a multivariate Pareto distribution model with Pareto type I distribution as its marginal distribution, and studied its probability density function, cumulative distribution function, and statistical properties[14]. Afterwards, Mardia and Arnold introduced the multivariate Pareto type II distribution based on the same statistical method. This distribution not only exhibits the heavy-tailed characteristics of the Pareto distribution but also can characterize the dependency among variables[15]. Lindley and Singpurwalla introduced the bivariate Lomax distribution, a special case of the bivariate Pareto type II distribution. They investigated its properties and applied it to model the lifetimes of two components operating under a familiar environment[16]. Asimit et al. proposed the MOBVPA distribution and employed the EM algorithm to estimate its unknown parameters[17]. Kumar Dey and Paul further refined the aforementioned EM algorithm to achieve more accurate parameter estimation in the MOBVPA model[18].

MOBVPA distribution works quite effectively to analyze data when some of the two components of the standardized dataset (location-scale transformation) take equal values because it has a singular part. However, in real-life datasets, we do not know the exact value of location and scale parameters, so it is quite impossible to know whether the standardized dataset has equal components. Therefore, the MOBVPA model with location and scale leads to misleading results when the standardized dataset does not have equal components. Based on Marshall-Olkin bivariate exponential distribution, Block and Basu proposed a bivariate exponential distribution that consists solely of an absolutely continuous part. They conducted an in-depth investigation into the probability density function, cumulative distribution function, and related statistical properties of this distribution[19]. Similar to Block and Basu's approach, Paul et al. further removed the singular part from the MOBVPA distribution, defining the Block-Basu bivariate Pareto (BBBVPA) distribution. This distribution provides a more flexible tool for analyzing bivariate data. Furthermore, Paul, Kumar Dey, and Kundu employed

Bayesian inference to estimate parameters in the BBBVPA distribution, without location and scale parameters[20]. In subsequent research, Paul and Kumar Dey introduced location and scale parameters into the BBBVPA model, further enhancing its flexibility. Compared to the multivariate Pareto type II distribution, the bivariate Lomax distribution, and the multivariate Pareto distribution with marginal Pareto type I distributions, the BBBVPA distribution's marginal distribution incorporates multiple parameters. This allows it to provide distinct shape characteristics for each variable, enabling more effective fitting to diverse datasets[21].

1.3 Research Content and Methods

To generalize the existing MOBVPA distribution, this study introduces proportional reversed hazard rate parameters to construct a generalized version of MOBVPA distribution. This model integrates location, scale, shape and proportional reversed hazard rate parameters, aiming to enhance the precision of fitting real-world data.

The structure of this paper is as follows. In Section 2, we first review the definitions and density functions of the classical Marshall-Olkin bivariate Pareto distribution and the Block-Basu bivariate Pareto distribution, along with key concepts in reliability analysis, including the reliability function, the reversed hazard rate function, and the reversed hazard rate parameter. In Section 3, we introduce a generalized form of the classical Marshall-Olkin Pareto distribution and its absolutely continuous version. For the generalized model, we derive expressions for the joint density function, marginal distributions, and conditional probability density functions. Furthermore, by comparing surface plots and contour plots of the joint density function of the generalized BBBVPA distribution under different parameter values, we identify the influence of the proportional reversed hazard rate parameter on the distribution's tail thickness, central tendency, and symmetry. Section 4 employs the EM algorithm to estimate parameters of the generalized Marshall-Olkin bivariate Pareto distribution. Furthermore, Monte Carlo simulations are used to evaluate the parameter estimation performance of EM algorithm under varying sample sizes.

2. Background

This chapter reviews the definitions and probabilistic structures of the classical Marshall-Olkin bivariate Pareto distribution, as well as Block and Basu bivariate Pareto distribution. Furthermore, we introduce key concepts in reliability analysis, including reliability functions and reversed hazard functions. Also, we revisit the definition of the reversed hazard rate parameter, providing a theoretical foundation for subsequent distribution extensions.

2.1 Marshall-Olkin Bivariate Pareto Distribution

First, we introduce the definition of the univariate Pareto Type II distribution.

Definition 1 If the probability density function of X is given by

$$f_{PA}(x;\mu,\sigma,\alpha)=\frac{\alpha}{\sigma}(1+\frac{x-\mu}{\sigma})^{-\alpha-1},x>\mu, \tag{1}$$

then X is said to follow Pareto type II distribution, where $\mu\in R,\sigma>0,\alpha>0$, denoted as $X\sim \mathrm{PA(II)}(\mu,\sigma,\alpha)$.

Also, it is easy to verify that, the survival function of $\mathrm{PA(II)}(\mu,\sigma,\alpha)$ is expressed as

$$S_{PA}(x;\mu,\sigma,\alpha)=(1+\frac{x-\mu}{\sigma})^{-\alpha},x>\mu. \tag{2}$$

The Marshall-Olkin bivariate Pareto distribution, as an extension of the univariate Pareto type II distribution, can characterize the dependence structure between two random variables, making it widely applicable in fields such as financial risk management, insurance actuarial science, and environmental science. Kumar Dey and Paul provided the following definition for the Marshall-Olkin bivariate Pareto distribution[20].

Definition 2 Suppose, as below, $U_0 \sim \mathrm{PA(II)}(0,1,\alpha_0)$, $U_1 \sim \mathrm{PA(II)}(\mu_1,\sigma_1,\alpha_1)$ and $U_2 \sim \mathrm{PA(II)}(\mu_2,\sigma_2,\alpha_2)$, and they are mutually independent random variables. Define $X_1 = \min\{\sigma_1 U_0 + \mu_1, U_1\}$ and $X_2 = \min\{\sigma_2 U_0 + \mu_2, U_2\}$, then the bivariate random variable (X_1, X_2) follows MOBVPA distribution with parameters $(\mu_1,\mu_2,\sigma_1,\sigma_2,\alpha_0,\alpha_1,\alpha_2)$, and it will be denoted as $(X_1,X_2) \sim \mathrm{MOBVPA}(\mu_1,\mu_2,\sigma_1,\sigma_2,\alpha_0,\alpha_1,\alpha_2)$.

Kumar Dey and Paul derived the following joint density function for the MOBVPA distribution[20].

Proposition 3 The joint probability density function of MOBVPA is

$$f(x_1,x_2)=\begin{cases} f_1(x_1,x_2), & \dfrac{x_2-\mu_2}{\sigma_2} > \dfrac{x_1-\mu_1}{\sigma_1} > 0; \\ f_2(x_1,x_2), & \dfrac{x_1-\mu_1}{\sigma_1} > \dfrac{x_2-\mu_2}{\sigma_2} > 0; \\ f_0(x), & \dfrac{x_1-\mu_1}{\sigma_1} = \dfrac{x_2-\mu_2}{\sigma_2} = x > 0. \end{cases} \tag{3}$$

Where

$$f_1(x_1,x_2)=\frac{\alpha_1(\alpha_0+\alpha_2)}{\sigma_1\sigma_2}\left(1+\frac{x_2-\mu_2}{\sigma_2}\right)^{-\alpha_0-\alpha_2-1}\left(1+\frac{x_1-\mu_1}{\sigma_1}\right)^{-\alpha_1-1},$$
$$f_2(x_1,x_2)=\frac{\alpha_2(\alpha_0+\alpha_1)}{\sigma_1\sigma_2}\left(1+\frac{x_2-\mu_2}{\sigma_2}\right)^{-\alpha_2-1}\left(1+\frac{x_1-\mu_1}{\sigma_1}\right)^{-\alpha_0-\alpha_1-1},$$
$$f_0(x)=\alpha_0(1+x)^{-\alpha_0-\alpha_1-\alpha_2-1}.$$

Kumar Dey and Paul employed the Block-Basu method to eliminate singular components of MOBVPA, defining the BBBVPA distribution as follows[20].

Definition 4 If the joint density function of a bivariate random variable (Y_1,Y_2) is

$$f_{BB}(y_1,y_2)=\begin{cases} cf_1(y_1,y_2), & \dfrac{y_2-\mu_2}{\sigma_2} > \dfrac{y_1-\mu_1}{\sigma_1} > 0; \\ cf_2(y_1,y_2), & \dfrac{y_1-\mu_1}{\sigma_1} > \dfrac{y_2-\mu_2}{\sigma_2} > 0, \end{cases} \tag{4}$$

where the normalization constant is

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$$c = \frac{\alpha_0 + \alpha_1 + \alpha_2}{\alpha_1 + \alpha_2}, \tag{5}$$

then the bivariate random variable (Y_1, Y_2) is said to follow Block-Basu bivariate Pareto distribution, denoted as BBBVPA $(\mu_1, \mu_2, \sigma_1, \sigma_2, \alpha_0, \alpha_1, \alpha_2)$.

2.2 Reliability Measures and Proportional Hazard Rate Parameters

(1) Reliability

Definition 5 The probability that a product performs its specified function under specified conditions within a specified time period t is called the reliability of the product, defined as

$$R(t) = \mathrm{P}(T > t), t > 0, \tag{6}$$

where T is the product's lifetime, and t is the specified time period.

The relationship between the reliability $R(t)$ and the cumulative distribution function $F(t)$ of T is given by

$$R(t) = 1 - F(t), t > 0. \tag{7}$$

(2) Reversed hazard rate function

The reversed hazard rate function describes the conditional probability that a hazard (or "fault") has occurred within a unit time period before a given time point, given that the hazard (or fault) has already occurred at that time point.

Definition 6 Let random variable T denote the lifetime of a product or a system under specified conditions, with its density function $f(t)$ and distribution function $F(t)$. The reversed hazard rate function $r(t)$ is defined as

$$r(t) = \frac{f(t)}{F(t)}, t > 0. \tag{8}$$

The reversed hazard rate function $r(t)$ of T can be equivalently expressed as

$$r(t) = \lim_{\Delta t \to 0} \frac{P(t - \Delta t \le T \le t \mid T \le t)}{\Delta t}, t > 0. \tag{9}$$

Finally, we introduce the proportional reversed hazard rate parameter.

Definition 7 Consider a random variable T with distribution function $F(x;\beta)$,where β is a parameter in the distribution. If for another distribution function $F_0(x)$, $F(x;\beta)$ can be expressed as

$$F(x;\beta) = F_0^{\beta}(x),\ x \ge 0, \tag{10}$$

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then β is called the proportional reversed hazard rate parameter of distribution $F(x;\beta)$, abbreviated as the PRH parameter, and $F_0(x)$ is the baseline distribution function.

It can be verified that the reversed hazard rate function of $F(x;\beta)$ is β times that of $F_0(x)$. This is also why β is called he proportional reversed hazard rate parameter.

3. Marshall-Olkin Bivariate Pareto Distribution with Proportional Reversed Hazard Rate Parameter

This chapter introduces a proportional reversed hazard rate parameter to propose a generalized form of the classical Marshall-Olkin bivariate Pareto distribution. For this generalized model, we derive expressions for its joint probability density function and survival function. Second, based on the Block-Basu method, we define an absolutely continuous version of the Marshall-Olkin bivariate Pareto distribution with a proportional reversed hazard rate parameter, and derive expressions for its joint density function, marginal density functions, and conditional distribution density functions. Furthermore, by comparing surface plots and contour plots of the joint density function of the extended BBBVPA distribution under different parameters, we reveal the influence of the proportional reversed hazard rate parameter on the distribution's tail thickness, central tendency, and symmetry.

3.1 Marshall-Olkin Bivariate Pareto Distribution with Proportional Reversed Hazard Rate Parameter

Next, we introduce the definition of a univariate Pareto type II distribution with a proportional reversed hazard rate parameter.

Definition 11 If a random variable X has probability density function

$$f(x;\mu,\sigma,\alpha,\beta)=\frac{\alpha\beta}{\sigma}[1-(1+\frac{x-\mu}{\sigma})^{-\alpha}]^{\beta-1}(1+\frac{x-\mu}{\sigma})^{-\alpha-1}, x>\mu, \tag{11}$$

then the random variable X is said to follow a Pareto type II distribution with a proportional reversed hazard rate parameter, where $\mu\in R, \sigma>0, \alpha>0, \beta>0$. It is denoted as $X\sim \text{PRH-PA(II)}(\mu,\sigma,\alpha,\beta)$.

The survival function is calculated as

$$S(x;\mu,\sigma,\alpha,\beta)=1-[1-(1+\frac{x-\mu}{\sigma})^{-\alpha}]^{\beta}, x>\mu, \tag{12}$$

and the distribution function is

$$F(x;\mu,\sigma,\alpha,\beta)=[1-(1+\frac{x-\mu}{\sigma})^{-\alpha}]^{\beta}, x>\mu. \tag{13}$$

The definition of the Marshall-Olkin bivariate Pareto distribution with proportional reversed hazard rate parameter is given below.

Definition 12 Let, as follows, random variables $U_0\sim\text{PRH-PA(II)}(0,1,\alpha_0,\beta_0)$, $U_1\sim\text{PRH-PA(II)}(\mu_1,\sigma_1,\alpha_1,\beta_1)$ and $U_2\sim\text{PRH-PA(II)}(\mu_2,\sigma_2,\alpha_2,\beta_2)$. Assume that U_0, U_1, U_2 are mutually independent random variables. Define $X_1=\min\{\sigma_1 U_0+\mu_1, U_1\}$, $X_2=\min\{\sigma_2 U_0+\mu_2, U_2\}$,

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then the bivariate random variable (X_1, X_2) follows the PRH-MOBVPA distribution with parameters $(\mu_1,\mu_2,\sigma_1,\sigma_2,\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2)$, and it will be denoted as $(X_1, X_2) \sim$ PRH-MOBVPA $(\mu_1,\mu_2,\sigma_1,\sigma_2,\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2)$.

Below, we derive the joint density function for PRH-MOBVPA distribution.

Proposition 13 The joint probability density function of PRH-MOBVPA distribution is

$$f(x_1,x_2)=\begin{cases} f_1(x_1,x_2), & \frac{x_2-\mu_2}{\sigma_2}>\frac{x_1-\mu_1}{\sigma_1}>0; \\ f_2(x_1,x_2), & \frac{x_1-\mu_1}{\sigma_1}>\frac{x_2-\mu_2}{\sigma_2}>0; \\ f_0(x), & \frac{x_1-\mu_1}{\sigma_1}=\frac{x_2-\mu_2}{\sigma_2}=x>0, \end{cases} \tag{14}$$

where

$$\begin{aligned} f_1(x_1,x_2) &= f(x_1;\mu_1,\sigma_1,\alpha_1,\beta_1)[f(x_2;\mu_2,\sigma_2,\alpha_0,\beta_0)S(x_2;\mu_2,\sigma_2,\alpha_2,\beta_2) \\ &\quad + f(x_2;\mu_2,\sigma_2,\alpha_2,\beta_2)S(x_2;\mu_2,\sigma_2,\alpha_0,\beta_0)], \\ f_2(x_1,x_2) &= f(x_2;\mu_2,\sigma_2,\alpha_2,\beta_2)[f(x_1;\mu_1,\sigma_1,\alpha_0,\beta_0)S(x_1;\mu_1,\sigma_1,\alpha_1,\beta_1) \\ &\quad + f(x_1;\mu_1,\sigma_1,\alpha_1,\beta_1)S(x_1;\mu_1,\sigma_1,\alpha_0,\beta_0)], \\ f_0(x) &= f(x;0,1,\alpha_0,\beta_0)S(x;0,1,\alpha_1,\beta_1)S(x;0,1,\alpha_2,\beta_2). \end{aligned}$$

Proof: We prove this by case analysis.

(1) When $\frac{x_2-\mu_2}{\sigma_2}>\frac{x_1-\mu_1}{\sigma_1}>0$, since U_0,U_1,U_2 are mutually independent, the joint probability density function of (X_1, X_2) is

$$\begin{aligned} S(x_1,x_2) &= \mathrm{P}(X_1>x_1,X_2>x_2) \\ &= \mathrm{P}(\min\{\sigma_1U_0+\mu_1,U_1\}>x_1,\min\{\sigma_2U_0+\mu_2,U_2\}>x_2) \\ &= \mathrm{P}(U_0>\max\{\frac{x_1-\mu_1}{\sigma_1},\frac{x_2-\mu_2}{\sigma_2}\},U_1>x_1,U_2>x_2) \\ &= \mathrm{P}(U_0>\max\{\frac{x_1-\mu_1}{\sigma_1},\frac{x_2-\mu_2}{\sigma_2}\})\mathrm{P}(U_1>x_1)\mathrm{P}(U_2>x_2) \\ &= S(\max\{\frac{x_1-\mu_1}{\sigma_1},\frac{x_2-\mu_2}{\sigma_2}\};0,1,\alpha_0,\beta_0)S(x_1;\mu_1,\sigma_1,\alpha_1,\beta_1)S(x_2;\mu_2,\sigma_2,\alpha_2,\beta_2) \\ &= S(\frac{x_2-\mu_2}{\sigma_2};0,1,\alpha_0,\beta_0)S(x_1;\mu_1,\sigma_1,\alpha_1,\beta_1)S(x_2;\mu_2,\sigma_2,\alpha_2,\beta_2). \end{aligned}$$

Taking the second mixed partial derivative yields

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$$
\begin{aligned}
f_1(x_1,x_2) &= \frac{\partial^2}{\partial x_1 \partial x_2} S(x_1,x_2) \\
&= \frac{\partial}{\partial x_1}\{-S(x_1;\mu_1,\sigma_1,\alpha_1,\beta_1)[\frac{1}{\sigma_2} f(\frac{x_2-\mu_2}{\sigma_2};0,1,\alpha_0,\beta_0) S(x_2;\mu_2,\sigma_2,\alpha_2,\beta_2) \\
&\quad + f(x_2;\mu_2,\sigma_2,\alpha_2,\beta_2) S(\frac{x_2-\mu_2}{\sigma_2};0,1,\alpha_0,\beta_0)]\} \\
&= f(x_1;\mu_1,\sigma_1,\alpha_1,\beta_1)[f(x_2;\mu_2,\sigma_2,\alpha_0,\beta_0) S(x_2;\mu_2,\sigma_2,\alpha_2,\beta_2) \\
&\quad + f(x_2;\mu_2,\sigma_2,\alpha_2,\beta_2) S(x_2;\mu_2,\sigma_2,\alpha_0,\beta_0)].
\end{aligned}
$$

(2) When $\frac{x_1-\mu_1}{\sigma_1} > \frac{x_2-\mu_2}{\sigma_2} > 0$, by analogy with the derivation in (1), the joint survival function of (X_1, X_2) is

$$
S(x_1,x_2) = S(\frac{x_1-\mu_1}{\sigma_1};0,1,\alpha_0,\beta_0) S(x_1;\mu_1,\sigma_1,\alpha_1,\beta_1) S(x_2;\mu_2,\sigma_2,\alpha_2,\beta_2).
$$

Taking the second-order mixed partial derivative yields

$$
\begin{aligned}
f_2(x_1,x_2) &= \frac{\partial^2}{\partial x_1 \partial x_2} S(x_1,x_2) \\
&= \frac{\partial}{\partial x_2}\{-S(x_2;\mu_2,\sigma_2,\alpha_2,\beta_2)[\frac{1}{\sigma_1} f(\frac{x_1-\mu_1}{\sigma_1};0,1,\alpha_0,\beta_0) S(x_1;\mu_1,\sigma_1,\alpha_1,\beta_1) \\
&\quad + f(x_1;\mu_1,\sigma_1,\alpha_1,\beta_1) S(\frac{x_1-\mu_1}{\sigma_1};0,1,\alpha_0,\beta_0)]\} \\
&= f(x_2;\mu_2,\sigma_2,\alpha_2,\beta_2)[f(x_1;\mu_1,\sigma_1,\alpha_0,\beta_0) S(x_1;\mu_1,\sigma_1,\alpha_1,\beta_1) \\
&\quad + f(x_1;\mu_1,\sigma_1,\alpha_1,\beta_1) S(x_1;\mu_1,\sigma_1,\alpha_0,\beta_0)].
\end{aligned}
$$

(3) When $\frac{x_1-\mu_1}{\sigma_1} = \frac{x_2-\mu_2}{\sigma_2} = x > 0$, the calculation yields

$$
\begin{aligned}
&\int_{\mu_2}^{\infty}\int_{\mu_1+\frac{\sigma_1(x_2-\mu_2)}{\sigma_2}}^{\infty} f_2(x_1,x_2)dx_1dx_2 + \int_{\mu_1}^{\infty}\int_{\mu_2+\frac{\sigma_2(x_1-\mu_1)}{\sigma_1}}^{\infty} f_1(x_1,x_2)dx_2dx_1 \\
&= \int_0^{\infty} S(x;0,1,\alpha_0,\beta_0)[f(x;0,1,\alpha_1,\beta_1)S(x;0,1,\alpha_2,\beta_2) + f(x;0,1,\alpha_2,\beta_2)S(x;0,1,\alpha_1,\beta_1)]dx \\
&= \int_0^{\infty} S(x;0,1,\alpha_0,\beta_0)d(-S(x;0,1,\alpha_1,\beta_1)S(x;0,1,\alpha_2,\beta_2)) \\
&= S(x;0,1,\alpha_0,\beta_0)(-S(x;0,1,\alpha_1,\beta_1)S(x;0,1,\alpha_2,\beta_2))\Big|_0^{\infty} \\
&\quad - \int_0^{\infty}(-S(x;0,1,\alpha_1,\beta_1)S(x;0,1,\alpha_2,\beta_2))d(S(x;0,1,\alpha_0,\beta_0))dx \\
&= 1 - \int_0^{\infty} f(x;0,1,\alpha_0,\beta_0)S(x;0,1,\alpha_1,\beta_1)S(x;0,1,\alpha_2,\beta_2)dx.
\end{aligned}
$$

Since $\int_{\mu_2}^{\infty}\int_{\mu_1+\frac{\sigma_1(x_2-\mu_2)}{\sigma_2}}^{\infty} f_2(x_1,x_2)dx_1dx_2+\int_{\mu_1}^{\infty}\int_{\mu_2+\frac{\sigma_2(x_1-\mu_1)}{\sigma_1}}^{\infty} f_1(x_1,x_2)dx_2dx_1+\int_0^{\infty} f_0(x)dx=1$, the expression for $f_0(x)$ is

$$f_0(x)=f(x;0,1,\alpha_0,\beta_0)S(x;0,1,\alpha_1,\beta_1)S(x;0,1,\alpha_2,\beta_2).$$

This completes the proof.

Next, we define the generalized BBBVPA distribution.

Definition 14 If a bivariate random variable (Y_1,Y_2) has joint density function

$$f_{PHR-BB}(y_1,y_2)=\begin{cases} cf_1(y_1,y_2), & \frac{y_2-\mu_2}{\sigma_2}>\frac{y_1-\mu_1}{\sigma_1}>0; \\ cf_2(y_1,y_2), & \frac{y_1-\mu_1}{\sigma_1}>\frac{y_2-\mu_2}{\sigma_2}>0, \end{cases} \tag{15}$$

where the normalization constant

$$c=[1-\int_0^{\infty} f_0(x)dx]^{-1}, \tag{16}$$

then the bivariate random variable (Y_1,Y_2) is said to follow a Block-Basu bivariate Pareto distribution with proportional reversed hazard rate parameter, denoted as $(Y_1,Y_2)\sim$ PRH-BBBVPA $(\mu_1,\mu_2,\sigma_1,\sigma_2,\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2)$.

3.2 Probabilistic Properties of the Block-Basu Bivariate Pareto Distribution

In this section, we derive the marginal and conditional probability density functions of the PRH-BBBVPA distribution and plot its joint density function as a surface map and contour plot under different parameters.

3.2.1 Marginal and Conditional Distributions

Proposition 15 If $(Y_1,Y_2)\sim$ PRH-BBBVPA $(\mu_1,\mu_2,\sigma_1,\sigma_2,\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2)$, then the marginal density functions of Y_1 and Y_2 are as follows

$$\begin{aligned} f_{Y_1}(y_1)=c\{&F(y_1;\mu_1,\sigma_1,\alpha_2,\beta_2)[f(y_1;\mu_1,\sigma_1,\alpha_0,\beta_0)S(y_1;\mu_1,\sigma_1,\alpha_1,\beta_1) \\ &+f(y_1;\mu_1,\sigma_1,\alpha_1,\beta_1)S(y_1;\mu_1,\sigma_1,\alpha_0,\beta_0)] \\ &+f(y_1;\mu_1,\sigma_1,\alpha_1,\beta_1)S(y_1;\mu_1,\sigma_1,\alpha_2,\beta_2)S(y_1;\mu_1,\sigma_1,\alpha_0,\beta_0)\}, y_1>\mu_1 \end{aligned} \tag{17}$$

And

$$\begin{aligned} f_{Y_2}(y_2)=c\{&F(y_2;\mu_2,\sigma_2,\alpha_1,\beta_1)[f(y_2;\mu_2,\sigma_2,\alpha_0,\beta_0)S(y_2;\mu_2,\sigma_2,\alpha_2,\beta_2) \\ &+f(y_2;\mu_2,\sigma_2,\alpha_2,\beta_2)S(y_2;\mu_2,\sigma_2,\alpha_0,\beta_0)] \\ &+f(y_2;\mu_2,\sigma_2,\alpha_2,\beta_2)S(y_2;\mu_2,\sigma_2,\alpha_1,\beta_1)S(y_2;\mu_2,\sigma_2,\alpha_0,\beta_0)\}, y_2>\mu_2, \end{aligned} \tag{18}$$

where c is the normalization constant.

Proof: Since $f_{Y_1}(y_1)$ and $f_{Y_2}(y_2)$ can be derived using similar methods, for simplicity, we only provide the derivation for $f_{Y_1}(y_1)$.

Define $G_1=\int_{\mu_2}^{\mu_2+\sigma_2\frac{y_1-\mu_1}{\sigma_1}} f_2(y_1,y_2)dy_2, G_2=\int_{\mu_2+\sigma_2\frac{y_1-\mu_1}{\sigma_1}}^{\infty} f_1(y_1,y_2)dy_2$, then we have

$$f_{Y_1}(y_1)=\int_{\mu_2}^{\infty} f_{PHR-BB}(y_1,y_2)dy_2=c[G_1+G_2]. \tag{19}$$

(1) Calculate G_1

$$\begin{aligned}
G_1&=\int_{\mu_2}^{\mu_2+\sigma_2\frac{y_1-\mu_1}{\sigma_1}} f_2(y_1,y_2)dy_2\\
&=[f(y_1;\mu_1,\sigma_1,\alpha_0,\beta_0)S(y_1;\mu_1,\sigma_1,\alpha_1,\beta_1)\\
&\quad+f(y_1;\mu_1,\sigma_1,\alpha_1,\beta_1)S(y_1;\mu_1,\sigma_1,\alpha_0,\beta_0)]\int_{\mu_2}^{\mu_2+\sigma_2\frac{y_1-\mu_1}{\sigma_1}} f(y_2;\mu_2,\sigma_2,\alpha_2,\beta_2)dy_2\\
&=F(y_1;\mu_1,\sigma_1,\alpha_2,\beta_2)[f(y_1;\mu_1,\sigma_1,\alpha_0,\beta_0)S(y_1;\mu_1,\sigma_1,\alpha_1,\beta_1)\\
&\quad+f(y_1;\mu_1,\sigma_1,\alpha_1,\beta_1)S(y_1;\mu_1,\sigma_1,\alpha_0,\beta_0)].
\end{aligned}$$

(2) Calculate G_2

$$\begin{aligned}
G_2&=\int_{\mu_2+\sigma_2\frac{y_1-\mu_1}{\sigma_1}}^{\infty} f_1(y_1,y_2)dy_2\\
&=f(y_1;\mu_1,\sigma_1,\alpha_1,\beta_1)\int_{\mu_2+\sigma_2\frac{y_1-\mu_1}{\sigma_1}}^{\infty}[f(y_2;\mu_2,\sigma_2,\alpha_0,\beta_0)S(y_2;\mu_2,\sigma_2,\alpha_2,\beta_2)\\
&\quad+f(y_2;\mu_2,\sigma_2,\alpha_2,\beta_2)S(y_2;\mu_2,\sigma_2,\alpha_0,\beta_0)]dy_2\\
&=f(y_1;\mu_1,\sigma_1,\alpha_1,\beta_1)S(y_1;\mu_1,\sigma_1,\alpha_2,\beta_2)S(y_1;\mu_1,\sigma_1,\alpha_0,\beta_0).
\end{aligned}$$

Substituting the result of G_1,G_2 into (19) yields the expression for $f_{Y_1}(y_1)$ as

$$\begin{aligned}
f_{Y_1}(y_1)=c\{&F(y_1;\mu_1,\sigma_1,\alpha_2,\beta_2)[f(y_1;\mu_1,\sigma_1,\alpha_0,\beta_0)S(y_1;\mu_1,\sigma_1,\alpha_1,\beta_1)\\
&+f(y_1;\mu_1,\sigma_1,\alpha_1,\beta_1)S(y_1;\mu_1,\sigma_1,\alpha_0,\beta_0)]\\
&+f(y_1;\mu_1,\sigma_1,\alpha_1,\beta_1)S(y_1;\mu_1,\sigma_1,\alpha_2,\beta_2)S(y_1;\mu_1,\sigma_1,\alpha_0,\beta_0)\}.
\end{aligned}$$

This completes the proof.

Using Definition 14 and Proposition 15, it is straightforward to obtain the density functions of the conditional distributions $Y_2 \mid Y_1=y_1$ and $Y_1 \mid Y_2=y_2$. Therefore, we state the following proposition directly.

Proposition 16 If $(Y_1,Y_2)\sim$ PRH-BBBVPA $(\mu_1,\mu_2,\sigma_1,\sigma_2,\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2)$, then the conditional density functions of $Y_2\,|\,Y_1=y_1$ and $Y_1\,|\,Y_2=y_2$ are as follows

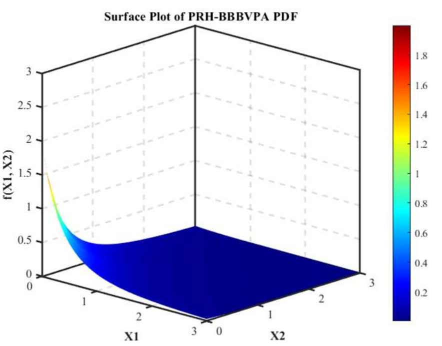
$$f_{Y_2|Y_1}(y_2|y_1)=\begin{cases} f_1(y_1,y_2)(cf_{Y_1}(y_1))^{-1}, & \dfrac{y_2-\mu_2}{\sigma_2}>\dfrac{y_1-\mu_1}{\sigma_1}>0; \\ f_2(y_1,y_2)(cf_{Y_1}(y_1))^{-1}, & \dfrac{y_1-\mu_1}{\sigma_1}>\dfrac{y_2-\mu_2}{\sigma_2}>0. \end{cases} \tag{20}$$

and

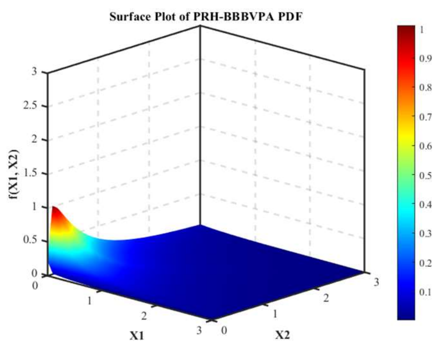
$$f_{Y_1|Y_2}(y_1|y_2)=\begin{cases} f_1(y_1,y_2)(cf_{Y_2}(y_2))^{-1}, & \dfrac{y_2-\mu_2}{\sigma_2}>\dfrac{y_1-\mu_1}{\sigma_1}>0; \\ f_2(y_1,y_2)(cf_{Y_2}(y_2))^{-1}, & \dfrac{y_1-\mu_1}{\sigma_1}>\dfrac{y_2-\mu_2}{\sigma_2}>0. \end{cases} \tag{21}$$

3.2.2 Surface Plot and Contour Plot of the Joint Density Function

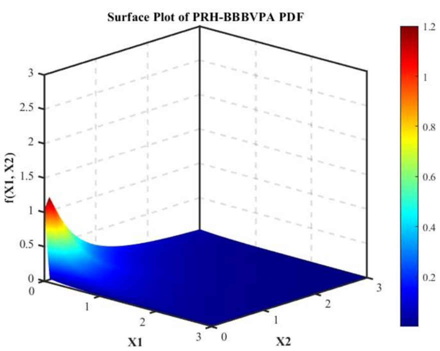
The surface plot and contour plot of the joint density function for the PRH-BBBVPA distribution are shown in Figures 1 and 2, respectively.



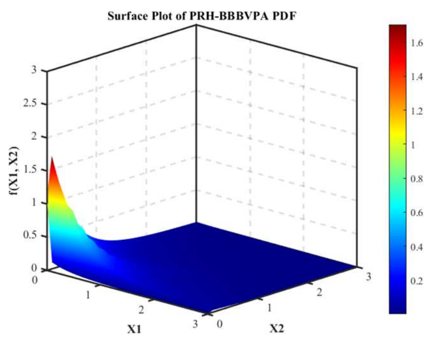
(a) PRH-BBBVPA(0,0,1,1,1,1,1,1,1,1)



(b) PRH-BBBVPA (0,0,1,1,1,1,1,1.2,1,1.4)



(c) PRH-BBBVPA (0,0,1,1,1,1,1,1,1,1.4)



(d) PRH-BBBVPA (0,0,1,1,1,1,2,1,1,1.4)

Figure 1. Surface plot of the joint density function for PRH-BBBVPA distribution

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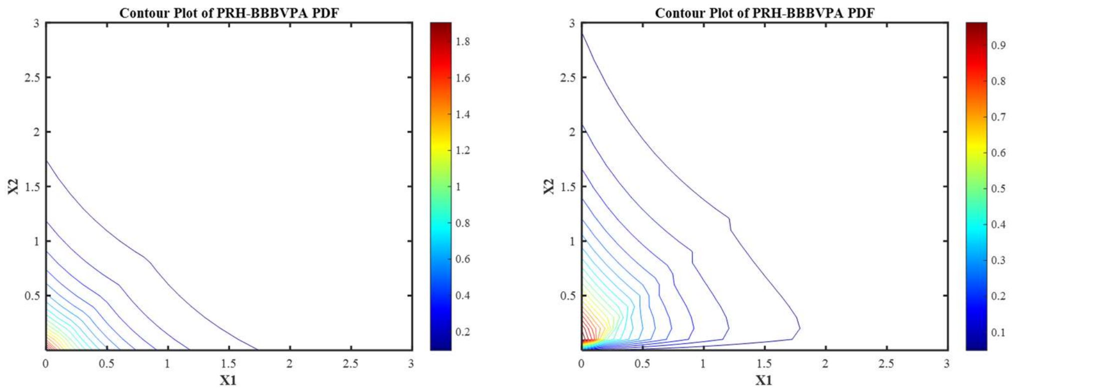
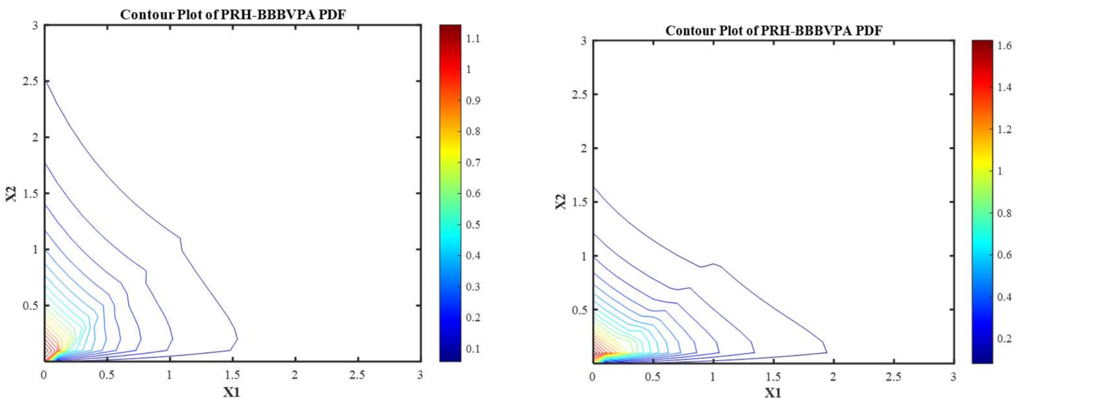


(a) PRH-BBBVPA (0,0,1,1,1,1,1,1,1,1) (b) PRH-BBBVPA(0,0,1,1,1,1,1,1.2,1,1.4)



(c) PRH-BBBVPA (0,0,1,1,1,1,1,1,1,1.4) (d) PRH-BBBVPA (0,0,1,1,1,1,2,1,1,1.4)

Figure 2. Contour plot of the joint density function for PRH-BBBVPA distribution

4. Parameter Estimators for PRH-MOBVPA Distribution

In this chapter, we will utilize the EM algorithm to explore the maximum likelihood estimators of parameters for the Marshall-Olkin bivariate Pareto distribution with proportional reversed hazard rate parameters. Monte Carlo simulations will be employed to evaluate the performance of the EM algorithm in estimating parameters for different sample sizes.

4.1 EM Algorithm

Now, assume $I=\{(x_{11},x_{21}),(x_{12},x_{22}),...,(x_{1n},x_{2n})\}$ is a random sample of size n from PRH-MOBVPA $(0,0,1,1,\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2)$. Divide the data into the following three classes.

$$I_0=\{i:x_{1i}=x_{2i}\},I_1=\{i:x_{1i}<x_{2i}\},I_2=\{i:x_{1i}>x_{2i}\}.$$

Define n_0,n_1,n_2 as $n_0=|I_0|,n_1=|I_1|,n_2=|I_2|$, where $|I_j|$ denotes the number of elements in the set I_j for $j=0,1,2$. Therefore, the log-likelihood function can be written as

$$\begin{aligned}
&\ln L(\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2)\\
&= n_0(\ln\alpha_0+\ln\beta_0)+n_1(\ln\alpha_1+\ln\beta_1)+n_2(\ln\alpha_2+\ln\beta_2)\\
&\quad+(\beta_0-1)\sum_{i\in I_0}\ln[1-(1+x_i)^{-\alpha_0}]-(\alpha_0+1)\sum_{i\in I_0}\ln(1+x_i)\\
&\quad+\sum_{i\in I_0}\ln[1-(1-(1+x_i)^{-\alpha_1})^{\beta_1}]+\sum_{i\in I_0}\ln[1-(1-(1+x_i)^{-\alpha_2})^{\beta_2}]\\
&\quad+(\beta_1-1)\sum_{i\in I_1}\ln[1-(1+x_{1i})^{-\alpha_1}]-(\alpha_1+1)\sum_{i\in I_1}\ln(1+x_{1i})\\
&\quad+\sum_{i\in I_1}\ln\{\alpha_0\beta_0[1-(1+x_{2i})^{-\alpha_0}]^{\beta_0-1}(1+x_{2i})^{-\alpha_0-1}[1-(1-(1+x_{2i})^{-\alpha_2})^{\beta_2}]\\
&\quad+\alpha_2\beta_2[1-(1+x_{2i})^{-\alpha_2}]^{\beta_2-1}(1+x_{2i})^{-\alpha_2-1}[1-(1-(1+x_{2i})^{-\alpha_0})^{\beta_0}]\}\\
&\quad+(\beta_2-1)\sum_{i\in I_2}\ln[1-(1+x_{2i})^{-\alpha_2}]-(\alpha_2+1)\sum_{i\in I_2}\ln(1+x_{2i})\\
&\quad+\sum_{i\in I_2}\ln\{\alpha_0\beta_0[1-(1+x_{1i})^{-\alpha_0}]^{\beta_0-1}(1+x_{1i})^{-\alpha_0-1}[1-(1-(1+x_{1i})^{-\alpha_1})^{\beta_1}]\\
&\quad+\alpha_1\beta_1[1-(1+x_{1i})^{-\alpha_1}]^{\beta_1-1}(1+x_{1i})^{-\alpha_1-1}[1-(1-(1+x_{1i})^{-\alpha_0})^{\beta_0}]\}.
\end{aligned} \tag{22}$$

As expected, the maximum likelihood estimators of $\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1$ and β_2 cannot be obtained in explicit form. Next, we utilize the EM algorithm to estimate the maximum likelihood parameters. According to the definition of PRH-BBBVPA, we know that (X_1,X_2) can be expressed as $(\min\{U_1,U_0\},\min\{U_2,U_0\})$. However, we do not know whether X_1 is U_0 or U_1, and similarly, we also have no information regarding X_2 is U_0 or U_2. Let us introduce a pair of random variables (Δ_1,Δ_2) associated with each (X_1,X_2) as

$$\Delta_1=\begin{cases}0\ ,\ X_1=U_0;\\1\ ,\ X_1=U_1,\end{cases}$$

$$\Delta_2=\begin{cases}0\ ,\ X_2=U_0;\\2\ ,\ X_2=U_2.\end{cases}$$

Table 1. Groups and corresponding orderings of hidden random variables U_0,U_1 and U_2.

Ordering	(X_1,X_2)	Group
$U_0<U_1<U_2$	(U_0,U_0)	I_0
$U_0<U_2<U_1$	(U_0,U_0)	I_0
$U_1<U_0<U_2$	(U_1,U_0)	I_1
$U_1<U_2<U_0$	(U_1,U_2)	I_1
$U_2<U_0<U_1$	(U_0,U_2)	I_2
$U_2<U_1<U_0$	(U_1,U_2)	I_2

According to Table 1, for the set I_0, $(\Delta_1,\Delta_2)=(0,0)$. For the set I_1, (Δ_1,Δ_2) may be (U_1,U_0) or (U_1,U_2). For the set I_2, (Δ_1,Δ_2) may be (U_0,U_2) or (U_1,U_2). Thus, (Δ_1,Δ_2) are not known for

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all the observations. To implement the EM algorithm, first, we obtain the E-step. In the E-step, the 'pseudo-log-likelihood' function is formed from the log-likelihood function by replacing the log-likelihood contribution of (X_1, X_2) by its expected value if the corresponding (Δ_1, Δ_2) is missing. In the M-step, we estimate the unknown parameters by maximizing this 'pseudo-log-likelihood' function with respect to the unknown parameters. The observations in Table 1 are used for constructing E-step.

To derive the 'pseudo-log-likelihood' function, we examine the conditional distributions of Δ_1 and Δ_2 under different groupings.

$$u_1 = \mathrm{P}\left(\Delta_2 = 0 \middle| I_1\right) = \frac{\mathrm{P}(U_1 < U_0 < U_2)}{\mathrm{P}(U_1 < U_0 < U_2) + \mathrm{P}(U_1 < U_2 < U_0)},$$
$$u_2 = \mathrm{P}\left(\Delta_2 = 2 \middle| I_1\right) = \frac{\mathrm{P}(U_1 < U_2 < U_0)}{\mathrm{P}(U_1 < U_0 < U_2) + \mathrm{P}(U_1 < U_2 < U_0)},$$
$$w_1 = \mathrm{P}\left(\Delta_1 = 0 \middle| I_2\right) = \frac{\mathrm{P}(U_2 < U_0 < U_1)}{\mathrm{P}(U_2 < U_0 < U_1) + \mathrm{P}(U_2 < U_1 < U_0)},$$
$$w_2 = \mathrm{P}\left(\Delta_1 = 1 \middle| I_2\right) = \frac{\mathrm{P}(U_2 < U_1 < U_0)}{\mathrm{P}(U_2 < U_0 < U_1) + \mathrm{P}(U_2 < U_1 < U_0)}.$$

Assume $F_{U_i}, S_{U_i}, f_{U_i}$ is the distribution function, survival function, and probability density function of U_i, respectively, and $i = 0,1,2$. Then we have

$$\begin{aligned}
&\mathrm{P}(U_1 < U_0 < U_2) \\
&= \int_0^\infty \mathrm{P}(U_1 < x < U_2 | U_0 = x) f_{U_0}(x) dx \\
&= \int_0^\infty \mathrm{P}(U_1 < x) \mathrm{P}(U_2 > x) f_{U_0}(x) dx \\
&= \int_0^\infty F_{U_1}(x) S_{U_2}(x) f_{U_0}(x) dx \\
&= \alpha_0 \beta_0 \int_0^\infty [1-(1+x)^{-\alpha_1}]^{\beta_1} [1-[1-(1+x)^{-\alpha_2}]^{\beta_2}] [1-(1+x)^{-\alpha_0}]^{\beta_0 - 1} (1+x)^{-\alpha_0 - 1} dx.
\end{aligned}$$

$$\begin{aligned}
&\mathrm{P}(U_1 < U_2 < U_0) \\
&= \int_0^\infty \mathrm{P}(U_1 < x < U_0 | U_2 = x) f_{U_2}(x) dx \\
&= \int_0^\infty \mathrm{P}(U_1 < x) \mathrm{P}(U_0 > x) f_{U_2}(x) dx \\
&= \int_0^\infty F_{U_1}(x) S_{U_0}(x) f_{U_2}(x) dx \\
&= \alpha_2 \beta_2 \int_0^\infty [1-(1+x)^{-\alpha_1}]^{\beta_1} [1-(1+x)^{-\alpha_2}]^{\beta_2 - 1} (1+x)^{-\alpha_2 - 1} [1-[1-(1+x)^{-\alpha_0}]^{\beta_0}] dx.
\end{aligned}$$

Similarly, we obtain

$$
\begin{aligned}
&\mathrm{P}(U_2 < U_0 < U_1) \\
&= \int_0^\infty \mathrm{P}(U_2 < x < U_1 | U_0 = x) f_{U_0}(x) dx \\
&= \int_0^\infty \mathrm{P}(U_2 < x)\mathrm{P}(U_1 > x) f_{U_0}(x) dx \\
&= \int_0^\infty F_{U_2}(x) S_{U_1}(x) f_{U_0}(x) dx \\
&= \alpha_0\beta_0 \int_0^\infty [1-(1+x)^{-\alpha_2}]^{\beta_2}[1-[1-(1+x)^{-\alpha_1}]^{\beta_1}][1-(1+x)^{-\alpha_0}]^{\beta_0-1}(1+x)^{-\alpha_0-1} dx,
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{P}(U_2 < U_1 < U_0) \\
&= \int_0^\infty \mathrm{P}(U_2 < x < U_0 | U_1 = x) f_{U_1}(x) dx \\
&= \int_0^\infty \mathrm{P}(U_2 < x)\mathrm{P}(U_0 > x) f_{U_1}(x) dx \\
&= \int_0^\infty F_{U_2}(x) S_{U_0}(x) f_{U_1}(x) dx \\
&= \alpha_1\beta_1 \int_0^\infty [1-(1+x)^{-\alpha_2}]^{\beta_2}[1-(1+x)^{-\alpha_1}]^{\beta_1-1}(1+x)^{-\alpha_1-1}[1-[1-(1+x)^{-\alpha_0}]^{\beta_0}] dx.
\end{aligned}
$$

Let $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2), \boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)$. Therefore, the 'pseudo-log-likelihood' function can be written as

$$
\begin{aligned}
Q = &\sum_{i \in I_0} \ln L(\boldsymbol{\alpha}, \boldsymbol{\beta} | x_i, \Delta_1 = 0, \Delta_2 = 0) \\
&+ \sum_{i \in I_1} [u_1 \ln L(\boldsymbol{\alpha}, \boldsymbol{\beta} | x_{1i}, x_{2i}, \Delta_1 = 1, \Delta_2 = 0) + u_2 \ln L(\boldsymbol{\alpha}, \boldsymbol{\beta} | x_{1i}, x_{2i}, \Delta_1 = 1, \Delta_2 = 2)] \\
&+ \sum_{i \in I_2} [w_1 \ln L(\boldsymbol{\alpha}, \boldsymbol{\beta} | x_{1i}, x_{2i}, \Delta_1 = 0, \Delta_2 = 2) + w_2 \ln L(\boldsymbol{\alpha}, \boldsymbol{\beta} | x_{1i}, x_{2i}, \Delta_1 = 1, \Delta_2 = 2)],
\end{aligned} \tag{23}
$$

where

$$
\begin{aligned}
&\ln L(\boldsymbol{\alpha}, \boldsymbol{\beta} | x_i, \Delta_1 = 0, \Delta_2 = 0) \\
&= \ln \beta_0 + (\beta_0 - 1)\ln[1-(1+x_i)^{-\alpha_0}] \\
&\quad + \ln \alpha_0 - (\alpha_0 + 1)\ln(1+x_i) + \ln\{1-[1-(1+x_i)^{-\alpha_1}]^{\beta_1}\} + \ln\{1-[1-(1+x_i)^{-\alpha_2}]^{\beta_2}\},
\end{aligned}
$$

$$
\begin{aligned}
&\ln L(\boldsymbol{\alpha}, \boldsymbol{\beta} | x_{1i}, x_{2i}, \Delta_1 = 1, \Delta_2 = 0) \\
&= \ln \beta_1 + (\beta_1 - 1)\ln[1-(1+x_{1i})^{-\alpha_1}] + \ln \alpha_1 - (\alpha_1 + 1)\ln(1+x_{1i}) \\
&\quad + \ln \beta_0 + (\beta_0 - 1)\ln[1-(1+x_{2i})^{-\alpha_0}] + \ln \alpha_0 - (\alpha_0 + 1)\ln(1+x_{2i}) + \ln\{1-[1-(1+x_{2i})^{-\alpha_2}]^{\beta_2}\},
\end{aligned}
$$

$$
\begin{aligned}
&\ln L(\boldsymbol{\alpha}, \boldsymbol{\beta} | x_{1i}, x_{2i}, \Delta_1 = 1, \Delta_2 = 2) \\
&= \ln \beta_1 + (\beta_1 - 1)\ln[1-(1+x_{1i})^{-\alpha_1}] + \ln \alpha_1 - (\alpha_1 + 1)\ln(1+x_{1i}) \\
&\quad + \ln \beta_2 + (\beta_2 - 1)\ln[1-(1+x_{2i})^{-\alpha_2}] + \ln \alpha_2 - (\alpha_2 + 1)\ln(1+x_{2i}) + \ln\{1-[1-(1+x_{2i})^{-\alpha_0}]^{\beta_0}\},
\end{aligned}
$$

$$
\begin{aligned}
&\ln L(\boldsymbol{\alpha}, \boldsymbol{\beta} | x_{1i}, x_{2i}, \Delta_1 = 0, \Delta_2 = 2) \\
&= \ln \beta_2 + (\beta_2 - 1)\ln[1-(1+x_{2i})^{-\alpha_2}] + \ln \alpha_2 - (\alpha_2 + 1)\ln(1+x_{2i}) \\
&\quad + \ln \beta_0 + (\beta_0 - 1)\ln[1-(1+x_{1i})^{-\alpha_0}] + \ln \alpha_0 - (\alpha_0 + 1)\ln(1+x_{1i}) + \ln\{1-[1-(1+x_{1i})^{-\alpha_1}]^{\beta_1}\},
\end{aligned}
$$

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$$\begin{aligned}&\ln L(\boldsymbol{\alpha},\boldsymbol{\beta}\,|\,x_{1i},x_{2i},\Delta_1=1,\Delta_2=2)\\&=\ln\beta_2+(\beta_2-1)\ln[1-(1+x_{2i})^{-\alpha_2}]+\ln\alpha_2-(\alpha_2+1)\ln(1+x_{2i})\\&\quad+\ln\beta_1+(\beta_1-1)\ln[1-(1+x_{1i})^{-\alpha_1}]+\ln\alpha_1-(\alpha_1+1)\ln(1+x_{1i})+\ln\{1-[1-(1+x_{1i})^{-\alpha_0}]^{\beta_0}\}.\end{aligned}$$

By differentiating with respect to the parameters $\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2$ and setting the derivative equal to zero, we can solve for the estimated values of the parameters. Clearly, its exact solution is challenging to obtain explicitly and approximate solutions can be computed via statistical software.

$$\begin{aligned}\frac{\partial Q}{\partial\alpha_0}&=\frac{n_0+u_1n_1+w_1n_2}{\alpha_0}+\sum_{i\in I_0}\frac{(\beta_0-1)(1+x_i)^{-\alpha_0}\ln(1+x_i)}{1-(1+x_i)^{-\alpha_0}}-\sum_{i\in I_0}\ln(1+x_i)\\&+\sum_{i\in I_1}\frac{u_1(\beta_0-1)(1+x_{2i})^{-\alpha_0}\ln(1+x_{2i})}{1-(1+x_{2i})^{-\alpha_0}}-\sum_{i\in I_1}u_1\ln(1+x_{2i})\\&-\sum_{i\in I_1}\frac{u_2\beta_0[1-(1+x_{2i})^{-\alpha_0}]^{\beta_0-1}(1+x_{2i})^{-\alpha_0}\ln(1+x_{2i})}{1-[1-(1+x_{2i})^{-\alpha_0}]^{\beta_0}}+\sum_{i\in I_2}\frac{w_1(\beta_0-1)(1+x_{1i})^{-\alpha_0}\ln(1+x_{1i})}{1-(1+x_{1i})^{-\alpha_0}}\\&-\sum_{i\in I_2}w_1\ln(1+x_{1i})-\sum_{i\in I_2}\frac{w_2\beta_0[1-(1+x_{1i})^{-\alpha_0}]^{\beta_0-1}(1+x_{1i})^{-\alpha_0}\ln(1+x_{1i})}{1-[1-(1+x_{1i})^{-\alpha_0}]^{\beta_0}},\end{aligned}\tag{24}$$

$$\begin{aligned}\frac{\partial Q}{\partial\alpha_1}&=\frac{n_1+w_2n_2}{\alpha_1}-\sum_{i\in I_0}\frac{\beta_1[1-(1+x_i)^{-\alpha_1}]^{\beta_1-1}(1+x_i)^{-\alpha_1}\ln(1+x_i)}{1-[1-(1+x_i)^{-\alpha_1}]^{\beta_1}}+\sum_{i\in I_1}\frac{(\beta_1-1)(1+x_{1i})^{-\alpha_1}\ln(1+x_{1i})}{1-(1+x_{1i})^{-\alpha_1}}\\&-\sum_{i\in I_1}\ln(1+x_{1i})-\sum_{i\in I_2}\frac{w_1\beta_1[1-(1+x_{1i})^{-\alpha_1}]^{\beta_1-1}(1+x_{1i})^{-\alpha_1}\ln(1+x_{1i})}{1-[1-(1+x_{1i})^{-\alpha_1}]^{\beta_1}}\\&+\sum_{i\in I_2}\frac{w_2(\beta_1-1)(1+x_{1i})^{-\alpha_1}\ln(1+x_{1i})}{1-(1+x_{1i})^{-\alpha_1}}-\sum_{i\in I_2}w_2\ln(1+x_{1i}),\end{aligned}\tag{25}$$

$$\begin{aligned}\frac{\partial Q}{\partial\alpha_2}&=\frac{n_2+u_2n_1}{\alpha_2}-\sum_{i\in I_0}\frac{\beta_2[1-(1+x_i)^{-\alpha_2}]^{\beta_2-1}(1+x_i)^{-\alpha_2}\ln(1+x_i)}{1-[1-(1+x_i)^{-\alpha_2}]^{\beta_2}}\\&-\sum_{i\in I_1}\frac{u_1\beta_2[1-(1+x_{2i})^{-\alpha_2}]^{\beta_2-1}(1+x_{2i})^{-\alpha_2}\ln(1+x_{2i})}{1-[1-(1+x_{2i})^{-\alpha_2}]^{\beta_2}}+\sum_{i\in I_1}\frac{u_2(\beta_2-1)(1+x_{2i})^{-\alpha_2}\ln(1+x_{2i})}{1-(1+x_{2i})^{-\alpha_2}}\\&-\sum_{i\in I_1}u_2\ln(1+x_{2i})+\sum_{i\in I_2}\frac{(\beta_2-1)(1+x_{2i})^{-\alpha_2}\ln(1+x_{2i})}{1-(1+x_{2i})^{-\alpha_2}}-\sum_{i\in I_2}\ln(1+x_{2i}),\end{aligned}\tag{26}$$

$$\begin{aligned}\frac{\partial Q}{\partial\beta_0}&=\frac{n_0+u_1n_1+w_1n_2}{\beta_0}+\sum_{i\in I_0}\ln[1-(1+x_i)^{-\alpha_0}]\\&+\sum_{i\in I_1}u_1\ln[1-(1+x_{2i})^{-\alpha_0}]-\sum_{i\in I_1}\frac{u_2[1-(1+x_{2i})^{-\alpha_0}]^{\beta_0}\ln[1-(1+x_{2i})^{-\alpha_0}]}{1-[1-(1+x_{2i})^{-\alpha_0}]^{\beta_0}}\\&+\sum_{i\in I_2}w_1\ln[1-(1+x_{1i})^{-\alpha_0}]-\sum_{i\in I_2}\frac{w_2[1-(1+x_{1i})^{-\alpha_0}]^{\beta_0}\ln[1-(1+x_{1i})^{-\alpha_0}]}{1-[1-(1+x_{1i})^{-\alpha_0}]^{\beta_0}},\end{aligned}\tag{27}$$

$$\begin{aligned}\frac{\partial Q}{\partial\beta_1}&=\frac{n_1+w_2n_2}{\beta_1}-\sum_{i\in I_0}\frac{[1-(1+x_i)^{-\alpha_1}]^{\beta_1}\ln[1-(1+x_i)^{-\alpha_1}]}{1-[1-(1+x_i)^{-\alpha_1}]^{\beta_1}}+\sum_{i\in I_1}\ln[1-(1+x_{1i})^{-\alpha_1}]\\&-\sum_{i\in I_2}\frac{w_1[1-(1+x_{1i})^{-\alpha_1}]^{\beta_1}\ln[1-(1+x_{1i})^{-\alpha_1}]}{1-[1-(1+x_{1i})^{-\alpha_1}]^{\beta_1}}+\sum_{i\in I_2}w_2\ln[1-(1+x_{1i})^{-\alpha_1}],\end{aligned}\tag{28}$$

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$$\frac{\partial Q}{\partial \beta_2}=\frac{u_2 n_1+n_2}{\beta_2}-\sum_{i\in I_0}\frac{[1-(1+x_i)^{-\alpha_2}]^{\beta_2}\ln[1-(1+x_i)^{-\alpha_2}]}{1-[1-(1+x_i)^{-\alpha_2}]^{\beta_2}}$$
$$-\sum_{i\in I_1}\frac{u_1[1-(1+x_{2i})^{-\alpha_2}]^{\beta_2}\ln[1-(1+x_{2i})^{-\alpha_2}]}{1-[1-(1+x_{2i})^{-\alpha_2}]^{\beta_2}}+\sum_{i\in I_1}u_2\ln[1-(1+x_{2i})^{-\alpha_2}] \tag{29}$$
$$+\sum_{i\in I_2}\ln[1-(1+x_{2i})^{-\alpha_2}].$$

4.2 Simulation

Table 2. Numerical simulation of PRH-MOBVPA$(0,0,1,1,\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2)$

n	parameters	AE	MSE	CI
25	α_0	1.8330823	0.0354911	[1.6197423, 2.0135324]
	α_1	0.6718722	0.0165817	[0.5772132, 1.1350589]
	α_2	0.7869933	0.1039942	[0.6632173, 0.9326835]
	β_0	0.9173673	0.0084772	[0.8077107, 1.0219337]
	β_1	0.7056671	0.0054254	[0.6183288, 0.8846498]
	β_2	0.7899814	0.0556331	[0.6671661, 0.8907023]
50	α_0	1.8491293	0.0289326	[1.6083360, 2.0155010]
	α_1	0.6495747	0.0080617	[0.5745017, 0.8001822]
	α_2	0.7724794	0.0320517	[0.6652194, 1.1136981]
	β_0	0.9049787	0.0073324	[0.7995944, 1.1115714]
	β_1	0.7139743	0.0047933	[0.6222173, 0.7833039]
	β_2	0.7620647	0.0064655	[0.6658466, 0.8975599]
75	α_0	1.8407584	0.0171114	[1.6184324, 2.0107181]
	α_1	0.6510841	0.0037100	[0.5744707, 0.7271717]
	α_2	0.7579460	0.0091863	[0.6634461, 0.8373368]
	β_0	0.8936428	0.0040844	[0.8009000, 1.0008796]
	β_1	0.7141792	0.0030300	[0.6193609, 0.7812861]
	β_2	0.7696028	0.0048908	[0.6728901, 0.8394030]
100	α_0	1.8239274	0.0138678	[1.6051118, 2.0084715]
	α_1	0.6383831	0.0026983	[0.5764971, 0.7208780]
	α_2	0.7439689	0.0032784	[0.6633182, 0.8351541]
	β_0	0.8935716	0.0028958	[0.8023589, 0.9938097]
	β_1	0.7121164	0.0022933	[0.6262672, 0.7833657]
	β_2	0.7631742	0.0024875	[0.6680688, 0.8370035]

To comprehensively evaluate the maximum likelihood estimates of the parameters for PRH-MOBVPA $(0,0,1,1,\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2)$ and the performance of the EM algorithm, we conducted a comprehensive simulation study. The average estimate (AE), mean squared error (MSE), and 95% percentile parametric bootstrap confidence interval (CI) are obtained for each parameter of PRH-MOBVPA. The AEs and MSEs are obtained for different sample sizes, namely n = 25, 50, 75, 100, based on 1000 replications or Monte Carlo (MC) runs. For each parameter, Cis are estimated based on a single MC run, with each run corresponding to 100 bootstrap samples. The actual parameter values were set to $\alpha_0 = 1.80, \alpha_1 = 0.65, \alpha_2 = 0.75,\ \beta_0 = 0.90,\ \beta_1 = 0.70, \beta_2 = 0.75$.

As shown in Table 2, with an increasing sample size, the mean estimates of the parameters approach the actual parameter values more closely, and the mean squared errors of the parameters decrease.

5. Conclusion

This paper investigates the Marshall-Olkin bivariate Pareto distribution with proportional reversed hazard rate parameters, as well as the Block-Basu bivariate Pareto distribution with proportional reversed hazard rate parameters. It derives expressions for related probabilistic properties, including the joint density function, joint survival function, marginal distributions, and conditional probability density functions. By comparing surface plots and contour plots of the joint density function of the PRH-BBBVPA distribution under different parameter values, the variation pattern of its joint density function with respect to these parameters is obtained. Additionally, in parameter estimation, the maximum likelihood estimates for the six parameters of the PRH-MOBVPA distribution are derived using the EM algorithm.

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