A New 3D Fractional-Order Chaotic System and its proof

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Abstract

In this paper, one new 3D fractional-order chaotic system with only one stable equilibrium is reported. To verify the chaoticity, the maximum Lyapunov exponent (MAXLE) with respect to the fractional-order and chaotic attractors are obtained by numerical calculation for this system. Numerical simulation results show that the chaotic attractor is emerged for the system when $0.958 \le q \le 1$.

Keywords

fractional-order chaotic system, equilibrium, Lyapunov exponent, numerical simulation

1. Introduction

Fractional-order calculus is an old branch of mathematics,which can be dated back to the 17th century [1, 2]. Now, it is well-known that many real-world physical systems [1–4] can be more accurately described by fractional-order differential equations, for example, dielectric polarization, viscoelasticity, electrode-electrolyte polarization, electromagnetic waves, diffusion-wave, superdiffusion, heat conduction.Meanwhile, chaotic behavior has been found in many fractional-order systems like the fractional-order brushless DC motor chaotic system [5,6], the fractional-order gyroscopes chaotic system[7], the fractional-order microelectromechanical chaotic system[8], the fractional-order electronic circuits [9, 10], and so forth [11–16].

Recently, a simple three-dimensional autonomous chaotic system [17] with only one stable node-focus equilibrium has been reported by Wang and Chen. Due to the impossibility of existence of homoclinic orbit and the unique stable node-focus equilibrium in this striking chaotic system, the well-known Si'lnikov criterions are not applicable. To verify the chaoticity in this system, Wang and Chen[17] calculated the largest Lyapunov exponent, fractional dimension, and continuous broad frequency spectrum by numerical calculation. Huan et al. presented a rigorous computer-assisted verification of horseshoe chaos by virtue of topological horseshoe theory [18]. Up to now, some integer order chaotic systems with stable node-focus equilibrium have been presented. To the best of our knowledge, many previous fractional-order chaotic systems like the fractional-order Lorenz chaotic system [19], the fractional-order Chen chaotic system [20], the fractional-order Lu chaotic system [21], the fractional-order brushless DC motor chaotic system [5, 6], the fractional-order gyroscopes chaotic system [7], the fractional-order microelectromechanical chaotic system [8], and so forth [9–14, 22, 23] have unstable equilibrium. There are seldom results on fractional-order chaotic systems with stable equilibrium. Hence, the finding of fractional-order chaotic systems with stable equilibrium is still an open problem.

Motivated by the above discussions, a three-dimensional autonomous fractional-order chaotic system with only one locally asymptotically stable equilibrium is proposed in this paper. The argument of all eigenvalues at equilibrium point satisfies $|\arg(\lambda_i)| \ge 0.5\pi (i = 1,2,3)$.Up to now, to the best of our knowledge, there are few results about the fractional-order chaotic systems with stable equilibrium. To verify the chaoticity in this fractional-order system, the maximum Lyapunov exponent and chaotic attractors are yielded by numerical calculation. The organization of this paper is as follows: in Section 2, a new fractional-order chaotic system with only one stable equilibrium is presented, and the maximum Lyapunov exponent and chaotic attractors are obtained. The conclusion is finally drawn in Section 3.

2. A New Fractional-Order Chaotic Systems with Only One Stable Equilibrium

In this paper, the q-order Caputo derivative for function $y(t)$ is defined as follows:

$$ {}_0^cD_t^q y(t) = \frac{1}{\Gamma(n-q)}\int_0^t \left[\frac{d^n y(\tau)}{d\tau^n}\right](t-\tau)^{n-q-1}d\tau \tag{1}$$

Where $n-1 \le q < n \in Z^+$ and $\Gamma(n-q) = \int_0^{+\infty} t^{n-q-1}e^{-t}dt$ is the Gamma function.

Now, we consider the following 3D nonlinear fractional-order system:

$$\begin{aligned} {}_0^cD_t^q x_1(t) &= \left(x_2(t) + \frac{1}{16}\right)x_3(t) \\ {}_0^cD_t^q x_2(t) &= x_1^{\ 2}(t) + 0.5x_1(t) - x_2(t) \\ {}_0^cD_t^q x_3(t) &= -2x_1(t) \end{aligned} \tag{2}$$

And, here, the fractional-order is $0 < q < 1$

First, let us recall the stability theorem for nonlinear commensurate fractional-order systems.

Lemma 1 (see [24, 25]). Consider the following fractional-order nonlinear system:

$$ {}_0^cD_t^q x\ (t) = g(x(t)) \tag{3}$$

Where $0 < q < 1$ is fractional-order, $x(t) \in R^n$ are state variables, and $g: R^n \to R^n$ is a nonlinear continuous vector function. The equilibrium point $\bar{x}\left(i.e., g\left(\bar{x}\right) = 0\right)$ in nonlinear fractional-order system (3) is locally asymptotically stable if $|\arg(\lambda_i(J))| \ge 0.5\pi q(i = 1,2,\ldots,n)$ here, J is the Jacobean matrix of $g(x(t))$ at equilibrium point $\bar{x}$ and $\lambda_i(i = 1,2,\ldots,n)$ are the eigenvalues of matrix J.

Now, we can obtain that system (2) has only one equilibrium point; that is, $\left(\bar{x}_1, \bar{x}_2, \bar{x}_3\right) = (0,0,0)$. Meanwhile, we can yield the eigenvalues of Jacobean matrix at equilibrium point as follows: $\lambda_1 = -1$, and $\lambda_\pm = \pm 0.25\sqrt{2}i$ So, we can obtain $|\arg(\lambda_i)| \ge 0.5\pi > 0.5\pi q(i = 1,2,3)$. According to the lemma, the equilibrium point $\left(\bar{x}_1, \bar{x}_2, \bar{x}_3\right) = (0,0,0)$ is locally asymptotically stable. Therefore, the equilibrium in system (2) is asymptotically stable.

Next, we discuss the numerical solution for system (2). Based on [20], we set $h = T/N$ and $t_n = nh(n = 0,1,2,\ldots N)$ and let initial condition be $(x_1(0), x_2(0), x_3(0))$. Soothe fractional-order system (2) can be discretized as follows:

$$x_1(n+1) = x_1(0) + \frac{h^q}{\Gamma(q+2)}\left\{\left[x_2^p(n+1) + \frac{1}{16}\right]\cdot x_3^p(n+1) + \sum_{j=0}^{n}\alpha_{1,j,n+1}\left[x_2(j) + \frac{1}{16}\right]x_3(j)\right\}$$

$$x_2(n+1)=x_2(0)+\frac{h^q}{\Gamma(q+2)}\left\{\left[\left(x_1^p(n+1)\right)^2+0.5x_1^p(n+1)-x_2^p(n+1)\right]+\sum_{j=0}^{n}\alpha_{2,j,n+1}\left[\left(x_1(j)\right)^2+0.5x_1(j)-x_2(j)\right]\right\}$$

$$x_3(n+1)=x_3(0)+\frac{h^q}{\Gamma(q+2)}\left\{\left[-2x_1^p(n+1)\right]+\sum_{j=0}^{n}\alpha_{3,j,n+1}\left[-2x_1(j)\right]\right\} \quad (4)$$

Where

$$x_1^p(n+1)=x_1(0)+\frac{1}{\Gamma(q)}\sum_{j=0}^{n}b_{1,j,n+1}\left[x_2(j)+\frac{1}{16}\right]x_3(j)$$

$$x_2^p(n+1)=x_2(0)+\frac{1}{\Gamma(q)}\sum_{j=0}^{n}b_{2,j,n+1}\left[\left(x_1(j)\right)^2+0.5x_1(j)-x_2(j)\right]$$

$$x_3^p(n+1)=x_3(0)+\frac{1}{\Gamma(q)}\sum_{j=0}^{n}b_{3,j,n+1}\left[-2x_1(j)\right] \quad (5)$$

$$\alpha_{i,j,n+1}=\begin{cases} n^{q+1}-(n-q)(n+1)^q, & j=0 \\ (n-j+2)^{q+1}+(n-j)^{q+1}-2(n-j+1)^{q+1}, & 1\le j\le n \\ 1, & j=n+1 \end{cases} \quad (i=1,2,3)$$

$$b_{i,j,n+1}=\frac{h^q}{q}\left[(n-j+1)^q-(n-j)^q\right],\quad 0\le j\le n$$

The error of this approximation is described as

$$\left|x_i(t_n)-x_i(n)\right|=o(h^p) \qquad (i=1,2,3),\ p=\min(2,1+q) \quad (6)$$

Now, some results are obtained by numerical calculation. Let $q=0.96$; Figure 1 show the results with initial condition $(x_1(0),x_2(0),x_3(0))=(-1.2,1,1)$.

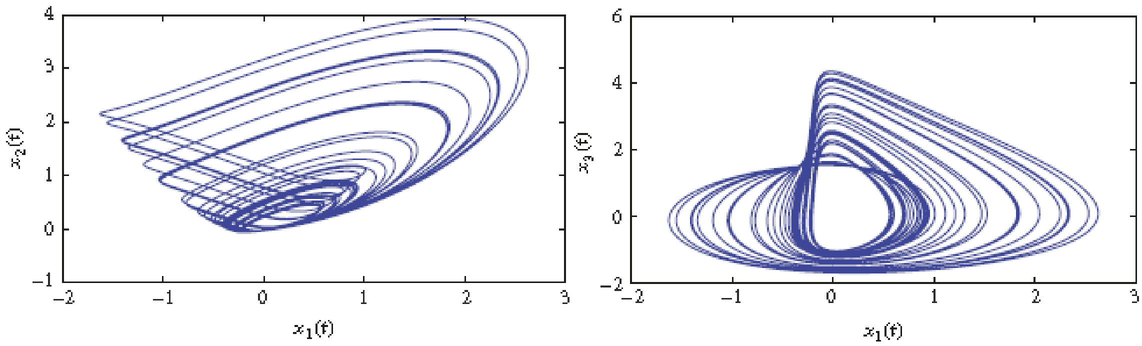

Fig. 1 A chaotic attractor in the system for $q=0.96$

The results in Figure 1 indicate that system (2) has a chaotic attractor if initial conditions are chosen as $(x_1(0),x_2(0),x_3(0))=(-1.2,1,1)$.

To verify the chronicity in system (2), we choose the initial conditions as $(x_1(0),x_2(0),x_3(0))=(-1.2,1,1)$ and calculate the maximum Lyapunov exponent (MAXLE) of system (2) with respect to the fractional-order q by numerical calculation. We obtain that the maximum Lyapunov exponent (MAXLE) is larger than zero for $0.958\le q\le 1$. Figure 2 shows the maximum Lyapunov exponent (MAXLE) varies as fractional-order q. So, the chaotic attractor is emerged in system (2) for $0.958\le q\le 1$.

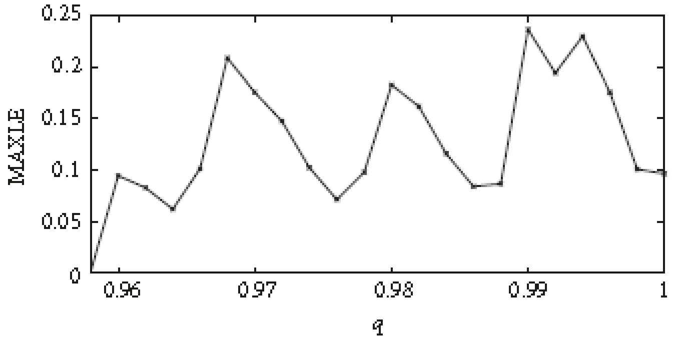

Fig. 2 The maximum Lyapunov exponent (MAXLE) varies as Fractional-order q

For example, the MAXLE is 0.0022 when $q = 0.958$, and its chaotic attractor is shown as Figure 3, while the MAXLE is 0.0946 when $q = 0.96$, and its chaotic attractor is shown as Figure 1.

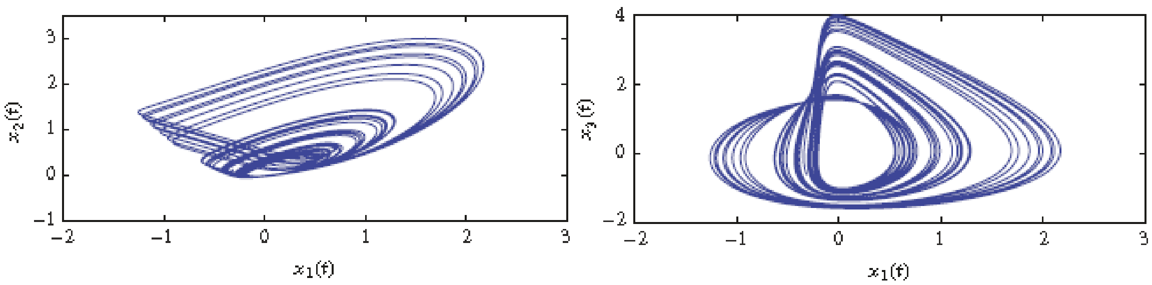

Fig. 3 A chaotic attractor in system (2) for $q = 0.958$

3. Conclusion

One new fractional-order chaotic system with only one stable equilibria point is reported in this paper. By numerical calculation, we yield the maximum Lyapunov exponent spectrum for this new fractional-order chaotic system, and the chaotic attractor can been found when $0.958 \le q \le 1$. The chaotic attractors for $q = 0.958$ and $q = 0.96$ are given. Of course, How to control the chaos of the system will be the next research direction.

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